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Dynamic Modeling and Load Sway Simulation of Jib Cranes

[1]*M.S. Al-Ajmi,* [2]*T.G. Abu-El Yazied,* [3]*S.Y. Abu-El-Hagag and* [4]*A.M. Makady*

[1,4]Department of Production Engineering Technology, College of Technological Studies, PAAET, Kuwait
[2]Department of Design and Production Engineering, Faculty of Engineering, Ain Shams University, Egypt
[3]Department of Structural Engineering, Faculty of Engineering, Ain Shams University, Cairo, Egypt

Abstract: In this work a nonlinear dynamic model representing load sway of jib cranes has been developed. The model considers the different motions of the crane. The model takes into account the simultaneous rotational motions of the boom and the post. These motions induce not only load sway in the plane of motion, but also oscillations of the plane determined by the ropes and the vertical axis through the suspension point. The data of an actual jib crane has been used to simulate the effect of transportation on the load sway. The simulation results illustrate that that the load sway may exceed the safe limits. Therefore, the developed model can be employed to achieve safe transportation planes. This model can be also used to apply active vibration control.

Key words: Jib Cranes · Load Sway · Dynamic Modeling · Simulation · Transportation Plan

INTRODUCTION

Jib cranes are widely used in workplaces to handle and transport heavy objects. The necessity to increase the speed of transportation generally induces undesirable sway of suspended objects and serious damage could occur during load handling and transportation. Therefore, it is necessary to predict the sway angles of the load by an accurate dynamical model such that safe transportation can be selected according to the developed model. This model can be also used to implement a satisfactory control scheme to suppress load sway. In the last few decades, several investigators considered the dynamic modeling of cranes [1-11]. The modeling problem was simplified by ignoring some of the crane motions [1-6]. In certain cases, the application of control strategies to these simplified models led to undesirable load sway [2]. In the work of Moustafa and Abu-El-Yazied [7], a time varying model of overhead crane load sway was developed. This model took into account simultaneous travel, transverse and hoisting motions. The stability of the above-developed model was also investigated. However, most of above work was, interested in overhead cranes and few works was interested with jib cranes [2, 3, 9, 11] which have completely different dynamic motions for the post, primary boom and secondary boom (Fig.1).

In this work, a nonlinear dynamic model of a jib crane is derived. The dynamics of the simultaneous post, boom and hoisting motions are taken into consideration. The derived model is implemented to simulate the response of a jib crane according to an actual transportation plan. The simulation results show the effectiveness of the derived model in the selection of appropriate transportation planes. This model can be also employed to apply an active vibration control which is capable to suppress the sway of the load.

Crane Total Energy: Jib crane is consisting of three links connected by three revolute pairs as shown in Fig. 1. The states of the crane motion will be considered as θ_i ; i = 1,2,3,4,5, where the first three angles represent the crane motion, while the last two states represents the sway angles of the load. The dynamic equations of the considered crane will be derived by the Lagrangian approach. The total kinetic energy of the crane can be obtained as follows:

$$KE = KE_1 + KE_2 + KE_3 + KE_4 \qquad (1)$$

The first three elements are the post; primary boom and secondary boom kinetic energy respectively, while the last term is the kinetic energy of the suspended mass.

Corresponding Author: Prof. Taher G. Abu-El-Yazied, Department of Design and Production Engineering, Faculty of Engineering, Ain Shams University, Cairo, Egypt. E-mail: taheryazied@yahoo.com

Fig. 1: Jib Crane

If the post, the primary boom and the secondary boom are considered to be of uniform cross section and mass per unit length and the length of each element i are denoted, respectively, by ρ_i and a_i, the first three terms can be derived to be:

$$
\begin{aligned}
KE_{1-3} &= KE_1 + KE_2 + KE_3 \\
&= \frac{1}{2} J_1 \dot{\theta}_1^{\,2} + \frac{1}{6} \rho_2 a_2^{\,3} \left(\dot{\theta}_2^{\,2} + \dot{\theta}_1^{\,2} \cos^2 \theta_2 \right) \\
&+ \frac{1}{2} \left(J_{11} \dot{\theta}_1^{\,2} + J_{22} \dot{\theta}_2^{\,2} + J_{33} \dot{\theta}_3^{\,2} + 2 J_{23} \dot{\theta}_2 \dot{\theta}_3 \right)
\end{aligned}
$$

Where

$$J_{11} = \frac{1}{3} \rho_3 a_3 (a_3^{\,2} c_{23}^2 + 3 a_2 a_3 c_2 c_{23} + 3 a_2^{\,2} c_2^{\,2}),$$

$$J_{22} = \frac{1}{3} \rho_3 a_3 \{ \left(a_3^{\,2} + 3 a_2^2 \right) + 3 a_2 a_3 c_3 \},$$

$$J_{33} = \frac{1}{3} \rho_3 a_3^3,$$

$$J_{23} = \frac{1}{3} \rho_3 a_3 \{ \left(a_3^{\,2} + 3 a_2^2 \right) + 3 a_2 a_3 c_3 \},$$

The load which is suspended at point "G", as shown in Fig.1, has a contribution to the kinetic and potential energies of the dynamic system. The crane motions θ_i ; = 1,2,3, are produced by actuators such as electrical motors. It is convenient to locate the inertia reference frame on the base, member "0". It can be shown that the position vector of point "D" (end effect) with respect to the reference frame (x_0, y_0, z_0), $R^0{}_D$ is as follows:

$$
R_D^0 = \begin{bmatrix} a_2 c_1 c_2 + a_3 c_1 c_{23} \\ a_2 s_1 c_2 + a_3 s_1 c_{23} \\ a_1 + a_2 s_2 + a_3 s_{23} \end{bmatrix} \tag{2}
$$

The details of the derivation of the position and displacement of the suspended load can be found in Appendix A. Accordingly, the position vector of payload "G" relative to "D" R_G^D it can be written as:

$$
R_G^D = \begin{bmatrix} s_4 c_{15} \\ -c_4 \\ -s_4 s_{15} \end{bmatrix} l \tag{3}
$$

Using equation 2-3, it is shown in Appendix A that the position of point "G" with respect to the inertia reference frame (x_0, y_0, z_0), is as follows:

$$
R_G^0 = \begin{bmatrix} l c_1 c_{23} s_4 c_{15} - (l c_4) s_{23} + l s_4 s_{15} c_{23} s_1 \\ -l s_4 c_{15} s_{23} c_1 - (l c_4) c_{23} - l s_4 s_{15} s_{23} s_1 \\ l s_4 c_{15} s_1 \end{bmatrix} \tag{4}
$$

Now the velocity vector of point "G" with respect to the fixed frame is obtained as follows:

$$
v_G = \begin{bmatrix} u \\ v \\ w \end{bmatrix}
$$

Where

$$
\begin{aligned}
v = l(s_{23} s_4 s_5) \dot{\theta}_1 + (a_2 c_3 + a_3) \dot{\theta}_2 + a_3 \dot{\theta}_3 \\
- l(s_{23} c_4 c_5 - c_{23} s_4) \dot{\theta}_4 + l(s_{23} s_4 s_5) \dot{\theta}_5
\end{aligned}
$$

$$u = -l(c_{23} s_4 s_5) \dot{\theta}_1 + a_2 s_3 \dot{\theta}_2 + l(c_{23} c_4 c_5 + s_{23} s_4) \dot{\theta}_4 - l(c_{23} s_4 s_5) \dot{\theta}_5$$

$$w = (a_2 c_2 + a_3 c_{23} - l s_4 c_5) \dot{\theta}_1 - l(c_4 s_5) \dot{\theta}_4 - l(s_4 c_5) \dot{\theta}_5$$

The kinetic energy of the suspended load can be written as follows:

$$
KE_4 = 0.5 \, M_G \left[u^2 + v^2 + w^2 \right] \tag{5}
$$

Therefore, the total kinetic energy of the crane can be written as follows:

$$KE = \frac{1}{2}J_1\dot{\theta}_1^{\,2} + \frac{1}{6}\rho_2 a_2^{\,3}\left(\dot{\theta}_2^{\,2} + \dot{\theta}_1^{\,2}\cos^2\theta_2\right) + \frac{1}{2}\left(J_{11}\dot{\theta}_1^{\,2} + J_{22}\dot{\theta}_2^{\,2} + J_{33}\dot{\theta}_3^{\,2} + 2J_{23}\dot{\theta}_2\dot{\theta}_3\right) + \frac{1}{2}M_G\left[u^2 + v^2 + w^2\right] \quad (6)$$

The potential energy of the crane can be written as:

$$PE = PE_1 + PE_2 + PE_3 + PE_4 \quad (7)$$

The first three elements are the post; primary boom and secondary boom potential energy respectively, while the last term is the potential energy of the suspended mass. The potential energy of the post is zero and therefore, we can write;

$$PE_{1-3} = PE_1 + PE_2 + PE_3 = \frac{1}{2}\rho_2\, g\, a_2^{\,2} s_2 + \rho_3 g\left(a_3 a_2 s_2 + \frac{1}{2}a^2{}_3 s_{23}\right) \quad (8)$$

Regarding the suspended load, of mass "M_G", the contribution to the potential energy can be derived as;

$$PE_4 = M_G g(a_2 s_2 + a_3 s_{23}) + M_G g l(1 - c_4 c_5) \quad (9)$$

Therefore, the total potential energy of the crane is;

$$PE = PE_{1\text{-}3} + PE_4 \quad (10)$$

Equations of Motion: The equations of motion for the crane based on Lagrangian dynamics can be obtained from;

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{\theta}_i}\right) - \frac{\partial KE}{\partial \theta_i} + \frac{\partial PE}{\partial \theta_i} = \tau_i \quad (11)$$

Where; I = 1,2,3,4,5 and τ_i, is the external torque applied at joint "i". It is obvious that the system energy contributions come from the energies of the post, boom, extended boom and the suspended load. The equations of motion associated with the generalized coordinates θ_i , i = 1,2,...,5 and can be obtained, respectively, as follows;

$$\sum_{i=1}^{5} A_{1i}\ddot{\theta}_i + \sum_{i,j=1,2}^{5} C_{1ij}(\theta_i,\theta_j)\dot{\theta}_i\dot{\theta}_j = \tau_1 \quad (12)$$

$$\sum_{i=1}^{5} A_{2i}\ddot{\theta}_i + \sum_{i,j=1,2}^{5} C_{2ij}(\theta_i,\theta_j)\dot{\theta}_i\dot{\theta}_j + D_{22} = \tau_2 \quad (13)$$

$$\sum_{i=1}^{5} A_{3i}\ddot{\theta}_i + \sum_{i,j=1,2}^{5} C_{3ij}(\theta_i,\theta_j)\dot{\theta}_i\dot{\theta}_j + D_{33} = \tau_3 \quad (14)$$

$$\sum_{i=1}^{5} A_{4i}\ddot{\theta}_i + \sum_{i,j=1,2}^{5} C_{4ij}(\theta_i,\theta_j)\dot{\theta}_i\dot{\theta}_j + D_{44} = 0 \quad (15)$$

$$\sum_{i=1}^{5} A_{5i}\ddot{\theta}_i + \sum_{i,j=1,2}^{5} C_{5ij}(\theta_i,\theta_j)\dot{\theta}_i\dot{\theta}_j + D_{55} = 0 \quad (16)$$

Numerical Simulation: To validate the derived mathematical model of the crane a simulation study will be conducted in this section. Usually the angular acceleration of the post, $\ddot{\theta}_1$ the angular accelerations of the primary boom and secondary boom, $\ddot{\theta}_2, \ddot{\theta}_3$, are decided beforehand according to the desired transportation plan. These motions are produced by the torques τ_1, τ_2, τ_3, of electrical motors attached to the respective joints. Therefore, the nominal torque of the different can be calculated equations 12-14 and by the nominal motions $\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3$. The data of the considered jib crane used in the simulation is illustrated in Table 1.

Table 1: Data of the Considered Crane

MG = 500*kg*.	$\rho_2 = \rho_3 = 1.5\ kg/m$.
a_2 = 15 *m*.	a_3 = 10 *m*.
l = 20 *m*.	J_1 = 10000 $kg.m^2$.

Equations 12-16 will be used to simulate the response of the considered crane for a specified transportation plan. The data of the considered crane are given in Table 1. The proposed transportation plan is to move the boom with a constant angular acceleration of 0.001 rad/sec^2 for twenty seconds. Then, the boom moves with a constant angular velocity for another forty seconds. Finally, the boom moves with a constant angular deceleration of 0.001 rad/sec^2 for another twenty seconds. The proposed transportation plan is illustrated on Fig.2 and the response is presented in Fig. 3-4. Since, in practice, the suspension ropes are not wound while the crane is in motion for safety considerations [4], no-hoisting of the load will be considered in the present study. Also, the motion of the base and the secondary boom are not considered. Fig. 4 shows the sway angle of the suspended load, θ_4 and it is obvious that the load exhibits persisting vibrations. It can also be noticed that the frequency of oscillation of θ_4 is about 0.1 Hz,

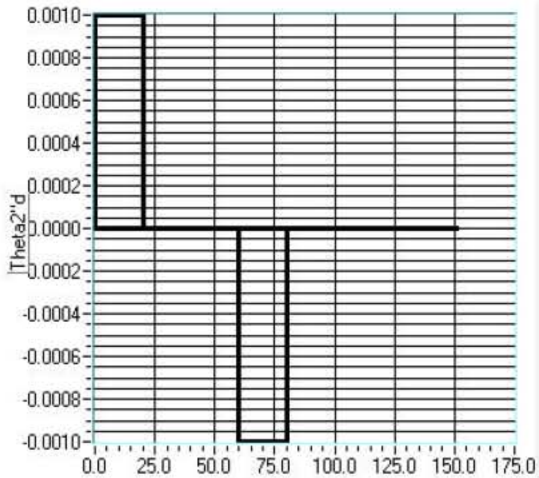

Fig. 2: Transportation plan “ $\ddot{\theta}_2$ ”

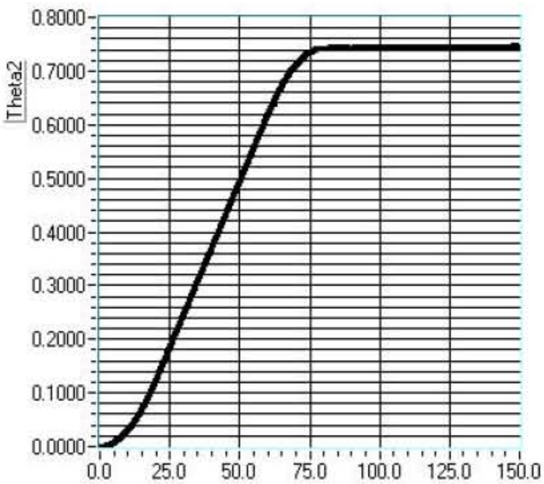

Fig. 3: Response of θ_2.

which agrees with the theoretical value obtained from the system equation. The results show the necessity of controlling the crane motion so that the load sway is reasonably suppressed. This will be considered in the next section.

Linearized Model: In order to make the developed model suitable to apply a control scheme to the above system, it is better to linearize the model about the normal operating conditions. These normal operating conditions can be calculated by the prescribed motions $\theta, \theta_2, \theta_3$ and by the nominal values of θ_4, θ_5 which can be calculated through equations 15 and 16. For instant, if the prescribed motion are constant acceleration or constant deceleration i.e.

$$\ddot{\theta}_1 = \ddot{\theta}_{10}, \quad \ddot{\theta}_2 = \ddot{\theta}_{20} \text{ and } \ddot{\theta}_3 = \ddot{\theta}_{30} \tag{17}$$

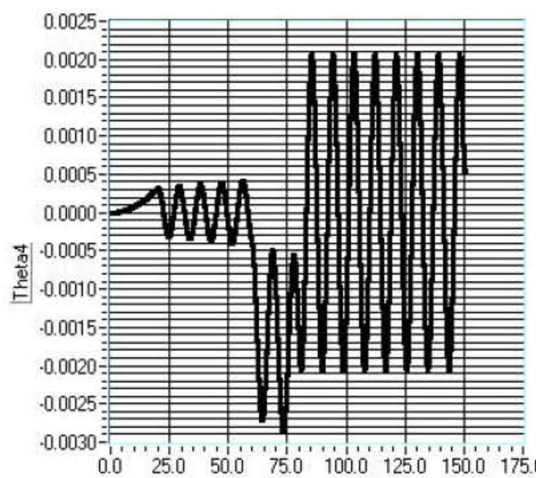

Fig. 4: Response of θ_4.

The nominal values of θ_4, θ_5 can be obtained by setting their derivatives equal to zero. Therefore, these values can be derived to be;

$$\theta_{40} = \sin^{-1}\left[\begin{array}{l} \sum_{i=1}^{3} A_{4i0}\ddot{\theta}_{i0} + \\ \sum_{i,j=1,2}^{3} C_{4ij0}(\theta_i,\theta_j)\dot{\theta}_{i0}\dot{\theta}_{j0} \end{array}\right] / \left(M_G g l c_{50}\right)$$

$$\theta_{50} = \sin^{-1}\left[\begin{array}{l} \sum_{i=1}^{3} A_{5i0}\ddot{\theta}_{i0} + \\ \sum_{i,j=1,2}^{3} C_{5ij0}(\theta_i,\theta_j)\dot{\theta}_{i0}\dot{\theta}_{j0} \end{array}\right] / \left(M_G g l c_{40}\right)$$

The case of small perturbation, about the nominal values, will be considered. Therefore, the linearized equations of motion can be written as follows;

$$\sum_{i=1}^{5} A_{1i0}\delta\ddot{\theta}_i + \sum_{i,j=1,2,}^{n} C_{1ij0}\delta\dot{\theta}_i\delta\dot{\theta}_j = \delta\tau_1 \tag{18}$$

$$\sum_{i=1}^{5} A_{2i0}\delta\ddot{\theta}_i + \sum_{i,j=1,2,}^{n} C_{2ij0}\delta\dot{\theta}_i\delta\dot{\theta}_j + D_{220}\delta\theta_2 = \delta\tau_2 \tag{19}$$

$$\sum_{i=1}^{5} A_{3i0}\delta\ddot{\theta}_{i1} + \sum_{i,j=1,2,}^{n} C_{3ij0}\delta\dot{\theta}_i\delta\dot{\theta}_j + D_{330}\delta\theta_3 = \delta\tau_3 \tag{20}$$

$$\sum_{i=1}^{5} A_{4i0}\delta\ddot{\theta}_i + \sum_{i,j=1,2,}^{n} C_{4ij0}\delta\dot{\theta}_i\delta\dot{\theta}_j + D_{440}\delta\theta_4 = 0 \tag{21}$$

$$\sum_{i=1}^{5} A_{5i0}\delta\ddot{\theta}_i + \sum_{i,j=1,2,}^{n} C_{5ij0}\delta\dot{\theta}_i\delta\dot{\theta}_j + D_{550}\delta\theta_5 = 0 \quad (22)$$

The case of plane motion will be considered in this section. The post will be considered fixed, i.e. $\theta_1 = 0$ and the relative motion between the secondary and primary booms will be ignored, i.e. $\theta_2 = \theta_3$. Therefore, the load sway angle θ_5 will be zero. In practice, the suspension ropes are not wound during crane motion, for safety considerations [5]. Hence, the hoisting of the load will be ignored, i.e. ($\dot{l}$= 0). Therefore, the dynamic equations of motion can be reduced to be:

$$\sum_{i=1}^{5} A_{2i0}\delta\ddot{\theta}_i + \sum_{i,j=1,2,}^{n} C_{2ij0}\delta\dot{\theta}_i\delta\dot{\theta}_j + D_{220}\delta\theta_2 = \delta\tau_2 \quad (24)$$

$$\sum_{i=1}^{5} A_{4i0}\delta\ddot{\theta}_i + \sum_{i,j=1,2,}^{n} C_{4ij0}\delta\dot{\theta}_i\delta\dot{\theta}_j + D_{440}\delta\theta_4 = 0 \quad (25)$$

CONCLUSION

A nonlinear dynamical model has been developed to represent the load sway dynamics of jib cranes, using the Lagrangian approach. This dynamical model considers the induced sway of the suspended load due to the simultaneous rotational motions of the boom, secondary boom and the post. This sway is induced not only in the plane of motion, but also in the plane determined by the ropes and the vertical axis through the suspension point. The derived model is adopted to simulate the response of an actual jib crane using a prescribed transportation plan. It is found that the load exhibits persistent oscillations for some transportation planes. The results show that the effectiveness of the derived model in selecting appropriate transportation planes. The developed is also linearized in order to apply a control scheme in a future work.

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APPENDIX A- Load Coordinates

The reference frame will be considered fixed to the base, member "0". The position vector of point "D" (end effect) with respect to the reference frame R_D^0 has been presented by equation 2. 0_j is located on point P_1 and x_3^j is directed along B_1P_1, as shown on Fig. A1. The vector R_G/p w.r.t. "j-reference frame (fixed in space) with origin o_j located at P_1 and x_3^j is directed along link 3 (B_1P_1) can be obtained. The position vector of point "G" relative to "D" R_G^D, it can be written as;

$$R_G^D = \begin{bmatrix} ls_4c_{15} \\ -lc_4 \\ -ls_4s_{15} \end{bmatrix} = \begin{bmatrix} s_4c_{15} \\ -c_4 \\ -s_4s_{15} \end{bmatrix} l \quad (A1)$$

In order to write energy expression, it is convenient to express the position of point "G" to the fixed reference frame (x_0, y_0, z_0), R_G^o as follows;

$$R_{G/D}^{i} = \begin{bmatrix} c_{23} & s_{23} & 0 \\ -s_{23} & c_{23} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & -s_1 \\ 0 & 1 & 0 \\ s_1 & 0 & c_1 \end{bmatrix} R_{G/P}^{j} = \begin{bmatrix} c_1 c_{23} & s_{23} & -c_{23} s_1 \\ -s_{23} c_1 & c_{23} & s_{23} s_1 \\ s_1 & 0 & 0 \end{bmatrix} R_{G/D}^{j} \quad \text{(A2)}$$

Equations (A1) and (A2) can be solved and get an expression of $R_{G/P}^{i}$.

$$R_G^0 = \begin{bmatrix} lc_1 c_{23} s_4 c_{15} - (lc_4) s_{23} + ls_4 s_{15} c_{23} s_1 \\ -ls_4 c_{15} s_{23} c_1 - (lc_4) c_{23} - ls_4 s_{15} s_{23} s_1 \\ ls_4 c_{15} s_1 \end{bmatrix} \quad \text{(A3)}$$

Velocity of the suspended load "M_G" can be written as;

$$\begin{aligned} R_g &= R_p + R_{G/P}^{i} \\ &= R_p + [R_x \underline{i_3} + R_y \underline{j_3} + R_z \underline{k_3}] \end{aligned} \quad \text{(A4)}$$

Where i3, j3, k3 are the unit vectors along the axis of the local reference frame ($X_3^i Y_3^i Z_3^i$). By differentiating eq. (A4), we get,

$$\begin{aligned} v_G = \dot{R}_G = \dot{R}_P &+ \left[\dot{R}_x \underline{i_3} + \dot{R}_y \underline{j_3} + \dot{R}_z \underline{k_3} \right] \\ &+ \left[R_x \frac{d\underline{i_3}}{dt} + R_y \frac{d\underline{j_3}}{dt} + R_z \frac{d\underline{k_3}}{dt} \right] \end{aligned} \quad \text{(A5)}$$

But the i- reference frame is rotating with angular velocity

$$\omega_{30} = \omega_1 \underline{i_3} + (\omega_2 + \omega_3) \underline{k_3} \quad \text{(A6)}$$

Substitution (6) in (5), we get

$$v_G = v_p + \left(\dot{R}_{G/P}^{i} \right)_{rel} + \left(\underline{\omega}_{30} \times R_{G/P}^{i} \right) \quad \text{(A7)}$$

Where $\underline{\omega}_{30}$ is the absolute angular velocity, which can be written as follows;

$$\underline{\omega}_{30} = (\omega_1 s_{23}) \underline{i}_3 + (\omega_1 c_{23}) \underline{j}_3 + (\omega_1 + \omega_3) \underline{k}_3 \quad \text{(A8)}$$

The following relations can be rewritten as follows;

$$R_{G/P}^{i} = R_x \underline{i}_3 + R_y \underline{j}_3 + R_z \underline{k}_3 \quad \text{(A9)}$$

$$\begin{aligned} \underline{v}_p &= (a_2 s_3 \omega_2) \underline{i_3} - [a_2 c_3 + a_2 c_{23}] \omega_1 \underline{k_3} \\ &+ [(a_2 c_3 + a_3) \omega_2 + a_3 \omega_3] \underline{j_3} \end{aligned} \quad \text{(A10)}$$

$$\underline{\omega}_{30} \times R_{G/P}^{i} = \begin{bmatrix} \underline{i_3} & \underline{j_3} & \underline{k_3} \\ \omega_1 s_{23} & \omega_1 c_{23} & \omega_2 + \omega_3 \\ R_x & R_y & R_z \end{bmatrix}$$

$$\begin{aligned} \underline{\omega}_{30} \times R_{G/P}^{i} &= [\omega_1 c_{23} R_z + (\omega_2 + \omega_3) R_y] \underline{i_3} \\ &- [\omega_1 s_{23} R_z + (\omega_2 + \omega_3) R_x] \underline{j_3} \\ &+ [\omega_1 s_{23} R_y + \omega_1 c_{23} R_x] \underline{k_3} \end{aligned}$$

Hence, the final expression for velocity vector of point "G" with respect to the fixed frame can be written as follows;

$$v_G = u \underline{i_3} + v \underline{j_3} + w \underline{k_3}$$

or in other words;

$$v_G = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Where

$$u = -l(c_{23} s_4 s_5)\dot{\theta}_1 + a_2 s_3 \dot{\theta}_2 + l(c_{23} c_4 c_5 + s_{23} s_4)\dot{\theta}_4 - l(c_{23} s_4 s_5)\dot{\theta}_5$$

$$\begin{aligned} v = l(s_{23} s_4 s_5)\dot{\theta}_1 &+ (a_2 c_3 + a_3)\dot{\theta}_2 + a_3 \dot{\theta}_3 \\ &- l(s_{23} c_4 c_5 - c_{23} s_4)\dot{\theta}_4 + l(s_{23} s_4 s_5)\dot{\theta}_5 \end{aligned}$$

$$w = (a_2 c_2 + a_3 c_{23} - l s_4 c_5)\dot{\theta}_1 - l(c_4 s_5)\dot{\theta}_4 - l(s_4 c_5)\dot{\theta}_5$$

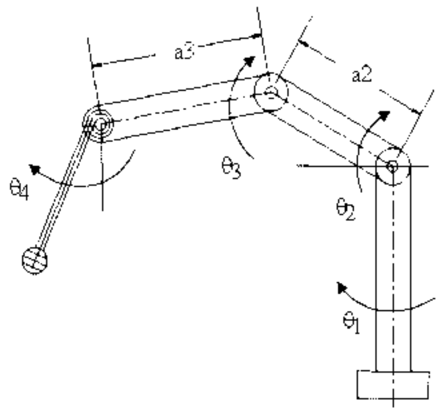

Fig. A1: Coordinates of the Crane.

APPENDIX B- Model Parameters:

$$A_{11} = \begin{bmatrix} J_1 + \frac{1}{3}\rho_2 a_2{}^3 c_2^2 + J_{11} + \\ M_G \begin{Bmatrix} l^2 (c_{23}s_4s_5)^2 + l^2 (s_{23}s_4s_5)^2 \\ +(a_2c_2 + a_3c_{23} - ls_4c_5)^2 \end{Bmatrix} \end{bmatrix}$$

$$A_{12} = A_{21} = M_G \begin{Bmatrix} -l(c_{23}s_4s_5)a_2s_3 \\ +l(s_{23}s_4s_5)(a_2c_3 + a_3) \end{Bmatrix}$$

$$A_{13} = A_{31} = MGla_3(S_{23}S_4S_5)$$

$$A_{14} = M_G \begin{Bmatrix} -l^2 (s_{23}s_4s_5)(s_{23}c_4c_5 - c_{23}s_4) \\ -l(c_4s_5)(a_2c_2 + a_3c_{23} - ls_4c_5) \end{Bmatrix}$$

$$A_{15} = M_G \begin{Bmatrix} l^2 (c_{23}s_4s_5)^2 + l^2 (s_{23}s_4s_5)^2 \\ -l(s_4c_5)(a_2c_2 + a_3c_{23} - ls_4c_5) \end{Bmatrix}$$

$$A_{22} = \begin{bmatrix} \frac{1}{3}\rho_2 a_2{}^3 + J_{22} \\ +M_G\left\{(a_2s_3)^2 + (a_2c_3 + a_3)^2\right\} \end{bmatrix}$$

$$A_{23} = A_{32} = J_{23} + MGa_3(a_2c_3 + a_3)$$

$$A_{24} = M_G \begin{bmatrix} a_2s_3l(c_{23}c_4c_5 + s_{23}s_4) \\ -l(a_2c_3 + a_3)(s_{23}c_4c_5 - c_{23}s_4) \end{bmatrix}$$

$$A_{25} = M_G \begin{bmatrix} -a_2s_3l(c_{23}s_4s_5) \\ +l(a_2c_3 + a_3)(s_{23}s_4s_5) \end{bmatrix}$$

$$A_{33} = \left[J_{33} + M_G a_3^2\right]$$

$$A_{34} = A_{43} = -M_G a_3 l(s_{23}c_4c_5 - c_{23}s_4)$$

$$A_{35} = A_{53} = MGa_3l(s_{23}s_4s_5)$$

$$A_{44} = M_G l^2 \begin{bmatrix} (c_{23}c_4c_5 + s_{23}s_4)^2 \\ +(s_{23}c_4c_5 - c_{23}s_4)^2 + (c_4s_5)^2 \end{bmatrix}$$

$$A_{45} = M_G l^2 \begin{bmatrix} -(c_{23}s_4s_5)(c_{23}c_4c_5 + s_{23}s_4) \\ +(s_{23}s_4s_5)(s_{23}c_4c_5 - c_{23}s_4) \\ +(s_4c_5)(c_4s_5) \end{bmatrix}$$

$$A_{55} = M_G l^2 \left[(c_{23}s_4s_5)^2 + (s_{23}s_4s_5)^2 + (s_4c_5)^2\right]$$

$$D_{11} = 0,$$

$$D_{22} = \frac{1}{2}\rho_2 g\, a_2{}^2 c_2 + \rho_3 g\left(a_3a_2c_2 + \frac{1}{2}a^2{}_3c_{23}\right) + M_G g(a_2c_2 + a_3c_{23})$$

$$D_{33} = \rho_3 g\left(\frac{1}{2}a^2{}_3 s_{23}\right) + M_G g a_3 c_{23}$$

$$D_{44} = MGgls_4c_5$$

$$D55 = MGglc_4s_5$$