On the Numerical Solutions of the Variable Order Fractional Heat Equation

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Abstract: In this paper, we consider the explicit finite-difference method for solving variable order fractional heat equation with a linear source term, via the Caputo concept of variable order fractional derivative. The stability of the method is discussed by means of the Gerschgorin theorem and using the stability matrix analysis. Numerical solutions of some variable order heat equation models are presented to demonstrate the effectiveness of the method.

Keywords: Variable order fractional derivatives, fractional heat equation, stability matrix analysis, Gerschgorin theorem

INTRODUCTION

While the fractional derivative a extension of integer order derivatives, the variable order fractional derivative is generalization of the constant derivative, when the order of derivative is a function. As we know, there are many definitions of the fractional derivative such as Riemann-Liouville, Grünwald-Letnikov and Caputo, definitions [1, 3, 4, 11-15]. On the other hand, there are many applications of the fractional derivatives in different fields, including biology, chemistry, physics, geology, mechanical engineering, electrical engineering, control theory, astrophysics and social sciences [1-5, 11, 12, 19, 20]. Samko and Ross (1993) first proposed the concept of variable order operator [13]. Lorenzo and Hartley (1998-2002) generalized different types of variable order fractional operator definitions and introduced some theoretical studies via the iterative Laplace transform [7-11]. As the fractional case, the variable order derivatives are useful in many fields, such as the control of nonlinear viscoelasticity oscillator [1, 2, 12]. The order can been a function changes with respect to time or space or another parameters [21]. Multiple definitions of a variable order derivative have been suggested [4, 6-8].

Definition: The Caputo space variable order derivative is defined as follows:

\[ D^{\alpha(x)}_tu(x,t)=\frac{1}{\Gamma(n-\alpha(x))}\int_0^t \frac{\partial^nu(x,\xi,t)}{\partial\xi^{n-\alpha(x)}}d\xi \]  

where \( n<\alpha(x)<n+1 \) [22].

Theorem (Gerschgorin theorem) [16]: Let the matrix \( A=(a_{nm}) \) has eigenvalues \( \lambda \) and define the absolute row and column sums respectively by:

\[ r_n = \sum_{k=0}^{N} |a_{nk}|, \quad c_m = \sum_{k=0}^{N} |a_{km}| \]

then, each eigenvalues lie in the union of the row circles \( R_n, n = 1,2,\ldots,N \), where

\[ R_n = \{ z : |z-a_m| \leq r_n \} \]

each eigenvalues lie in the union of the column circles \( C_m, m = 1,2,\ldots,N \), where

\[ C_m = \{ z : |z-a_m| \leq c_m \} \]

In this paper we use the FDM for studying the following linear space variable order heat equation:

\[ \frac{\partial u(x,t)}{\partial t} = B(x,t)\frac{\partial^{\alpha(x)}u(x,t)}{\partial x^{\alpha(x)}} + f(u(x,t),1<\alpha(x,t)\leq 2 \]  

with initial conditions

\[ u(x,0) = \varphi(x) \]

and the boundary conditions

\[ u(0,t) = \psi_1(x), \quad u(a,t) = \psi_2(x) \]

where \( 0\leq x \leq a, 0<t \leq T \) and \( f \) is linear scour term.

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In this section we will use the explicit finite difference method to study the model problem (2), then the space-time solutions domain will be discretized. The discrete form for the Caputo derivative (1) can be written as follows [17]:

\[ D_x^{\alpha(x,t)} u(x,t) = \frac{1}{\Gamma(2-\alpha(x,t))} \int_0^t (x-\xi)^{1-\alpha(x,t)} \frac{\partial^2 u(\xi,t)}{\partial \xi^2} d\xi \]

\[ = \frac{1}{\Gamma(2-\alpha(x,t))} \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{\partial^2 u(x,z,t)}{\partial z^2} dz, \quad z = x - \xi \]

\[ \approx \frac{1}{\Gamma(2-\alpha(x,t))} \sum_{i=0}^{k-1} \frac{u(x-(k-1)h,t) - 2u(x-kh,t) + u(x+(k-1)h,t)}{h^2} \times \int_{x_i}^{x_{i+1}} z^{1-\alpha(x,t)} dz \]

Then

\[ D_x^{\alpha(x,t)} u(x,t) = \frac{h^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))} \sum_{i=0}^{k-1} \frac{u(x-(k-1)h,t) - 2u(x-kh,t) + u(x+(k-1)h,t)}{h^2} \times ((k+1)^{\alpha(x,t)} - k^{\alpha(x,t)}) \]

Now, pick two positive integers N, M and define the step size of space and time, respectively, by h, \( \tau \) where

\[ h = \frac{a}{M} \quad \text{and} \quad \tau = \frac{T}{N} \]

Also we introduce the following notations:

\[ x_i = ih, \quad i = 1, 2, \ldots, M, \quad t_j = j\tau, \quad j = 1, 2, \ldots, N \]

\[ u^j_i = u(x_i, t_j), \quad \text{and} \quad f^j_i = f(x_i, \chi, \xi, \tau) \]

By using the forward difference to approximate \( \frac{\partial u(x,t)}{\partial t} \), we can rewrite equation (2) in the following form:

\[ \frac{u_{i+1}^j - u_i^j}{\tau} = \frac{B h^{-\alpha(i,j)}}{\Gamma(3-\alpha(i,j))} \sum_{i=0}^{k-1} (u_{i+1}^{j+1} - 2u_{i+1}^j + u_{i+1}^{j+1})((k+1)^{2-\alpha(i,j)} - k^{2-\alpha(i,j)}) + f_i^j \]

For simplicity let us define:

\[ R_i^j = \frac{\tau B h^{-\alpha(i,j)}}{\Gamma(3-\alpha(i,j))}, \quad G_i^j = ((i+1)^{2-\alpha(i,j)} - i^{2-\alpha(i,j)}) \]

then, we have

\[ u_{i+1}^j = u_i^j + R_i^j (u_{i+1}^j - 2u_i^j + u_{i-1}^j)G_i + \tau f_i^j \]

\[ u_{i+1}^j = u_i^j + R_i^j (u_{i+1}^j - 2u_i^j + u_{i-1}^j)G_i + \tau f_i^j \]

System (4) can be written at the level time \( t = t_j \) in the following matrix form

\[ U^{j+1} = A U^j + F^j \]

where

\[ F^j = \left( \tau f(u_{i-1}^j, x_{i-1}^j, t_i), \ldots, \tau f(u_i^j, \chi, \xi) \right)^T \]

and \( A^j = (a_{nm}) \) is a matrix with the following coefficients:

\[ a_{nm} = \begin{cases} 0, & \text{when } m > n + 2, \cr R_n G_{m+1}, & \text{when } m = n + 1, \cr R_n (G_{m+1} - 2G_m), & \text{when } m = n, \cr R_n (G_m - 2G_n) + 2, & \text{when } m = n - 1, \cr R_n (G_n - 2G_m), & \text{when } m = n - 2, \cr R_n (G_n - 2G_{m+2}), & \text{when } m = n - 3, \ldots, K, \cr R_n (G_n - 2G_{m+2}), & \text{when } m = n - 2, \cr \end{cases} \]
when \( R_n \) and \( G_n \), denote to \( R_n^j \) and \( G_n^j \), where \( n, m = 1, 2, \ldots, k - 1 \).

**THE STABILITY ANALYSIS AND THE TRUNCATION ERROR**

**Lemma 1:** Let \( A \in \mathbb{C}^{n \times n} \) and \( \rho(A) \) is the spectral radius of the matrix \( A \), then for any given positive number \( \varepsilon \), there exists a norm \( \| \cdot \| \) of the matrix \( A \) such that

\[
\| A \| \leq \rho(A) + \varepsilon + \varepsilon
\]

**Proof:** [18].

**Theorem 2:** If

\[
\max_{2 \leq n \leq K} \left| 1 - R_n \right|, \left| 1 - R_n (G_2 G_n) \right|, \left| 1 - R_n (1 - 2G_n) \right| \leq \frac{1}{2}
\]

then the finite difference scheme in (5) is stable.

**Proof:** From Gerschgorin theorem, the eigenvalues of the matrix \( A \) lie in the union of the \( N \) circles centered at \( (a_{nn}) \), with radius \( 2 \sqrt{a_{nn}} \):

\[
\sum_{k=0}^{N} r_{nn} \Gamma(2, x(t)) \Gamma(2, (x(t))) \frac{\partial^s u(\xi, t)}{\partial \xi^s} d\xi
\]

be a smooth function then

\[
\left| D_s^{(\alpha, 1)} u(\chi, \xi) - D_s^{(\alpha, 1)} u(\chi, \xi) \right| = O(h)
\]

**Proof:** [17].

From lemma 2, truncation error of explicit finite difference scheme (5) is \( O(h + \tau) \).

**NUMERICAL TEST EXAMPLES**

**Example 1:** Let us consider a variable-order linear fractional wave equation:

\[
\frac{\partial u(x,t)}{\partial t} = \alpha(x,t) A \frac{\partial^s u(\xi, t)}{\partial \xi^s} d\xi
\]

with

\[
\alpha(x, t) = 1.5 + 0.5 e^{-|x| - 1} \quad 0 \leq x \leq 8, \quad T = 1
\]

\[
f(u, x, t) = \frac{u}{80(t + 1)}
\]

Then, if

\[
\max_{2 \leq n \leq K} \left| 1 - R_n \right|, \left| 1 - R_n (G_2 G_n) \right|, \left| 1 - R_n (1 - 2G_n) \right| \leq \frac{1}{2}
\]

the spectral radius of the matrix \( A \) satisfy \( \rho(A) \leq 1 \), then from lemma 1, for any given positive number \( \varepsilon \), there exists a norm \( \| \cdot \| \) of the matrix \( A \) such that \( \| A \| \leq 1 + \varepsilon \).

Let \( W^{j+1} \) and \( U^{j+1} \) be two different numerical solutions of (5) with initial values given by \( W^0 \) and \( U^0 \), respectively and \( W^{j+1} - U^{j+1} = \varepsilon^{j+1} \).

Then

\[
\left\| \varepsilon^{j+1} \right\| \leq \left\| A \right\| \left\| \varepsilon^j \right\| \leq (1 + \varepsilon) \left\| \varepsilon^j \right\| \leq (1 + \varepsilon)^{j+1} \left\| \varepsilon^0 \right\| \leq C \left\| \varepsilon^0 \right\|
\]

where \( C \) is a constant.

Then the proof is complete.

**Lemma 2:** Let

\[
\Gamma(\alpha(x,t)) \Gamma(\alpha(\xi,t)) \frac{\partial^s u(\xi, t)}{\partial \xi^s} d\xi
\]

be a smooth function then

\[
\left| D_s^{(\alpha, 1)} u(\chi, \xi) - D_s^{(\alpha, 1)} u(\chi, \xi) \right| = O(h)
\]

**Proof:** [17].

From lemma 2, truncation error of explicit finite difference scheme (5) is \( O(h + \tau) \).
Table 1:

<table>
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<tr>
<th>x (t=0.25)</th>
<th>Exact</th>
<th>Approx</th>
<th>Error</th>
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Fig. 1:

and \( u(x,0) = \varphi(x) = x^2(8-x) \)

This problem has exact solution

\[ u(x,t) = \frac{1}{80}x^2(8-x)(t+1) \]

In the Table 1, a comparison between the numerical and the exact solutions using FDM at \( t = 0.25 \) is given:
Figure 1 shows the results in Table 1, where the approximate solution satisfies the stability condition. Moreover, if we chose $h, \tau$ such that

$$\max_{2 \leq k \leq K} |1-R_1|, |1-R_2(G_1-2G_2)|, |1-R_3(1-2G_1+G_2)| > \frac{1}{2}$$

then, the stability condition is not satisfied and the results showed in Fig. 2. In this case, the numerical solution is unstable.

Fig. 3: Shows the numerical solution in 3-D, while Fig. 4 shows the numerical solution changes with respect $\alpha(x,t)$

**Example 2:** Consider a variable-order linear fractional wave equation:

$$\frac{\partial}{\partial t} u(x,t) = \frac{x^2}{6} D^\alpha_{x,t} u(x,t) - 4xe^t$$

with

$$\alpha(x,t) = 1.25 + 0.25\cos(x)\sin(2t)$$

and

$$u(x,0) = \varphi(x) = x(x^2 - 4), \quad u(0,t)u(2,t) = 0$$

where $0 \leq x \leq 2, \quad T = 1$.

In case $\alpha = 2$, the above problem has the following exact solution:

$$u(x,t) = xe^t(x^2 - 4)$$

In Fig. 5, a comparison between the numerical solution when $\alpha = \alpha(x,t)$ and the exact solution when $\alpha = 2$, at $t = 0.45$.

In Fig. 6, the numerical solution at all values of the time and Fig. 7 shows the numerical solution changes with respect $\alpha(x,t)$

**CONCLUSIONS**

In this paper, we used the finite difference methods with Caputo estimates for solving the variable order heat equation, also we prove the condition of the stability and determined the truncation error of the method in explicit method. Some test examples are given, we can say that this method in its general form gives a reasonable calculations, easy to use and can be applied for the variable order differential equations. All results obtained by using MATLAB version 7.6.0(R2008a).
REFERENCES