Numerical Solution of Nonlinear Multi-order Fractional Differential Equations
by Implementation of the Operational Matrix of Fractional Derivative

M.M. Khader

Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

Abstract: The main aim of this article is to generalize the Legendre operational matrix to the fractional derivatives and implemented it to solve the nonlinear multi-order fractional differential equations. In this approach, a truncated Legendre series together with the Legendre operational matrix of fractional derivatives are used. The main characteristic behind the approach using this technique is that it reduces such problems to those of solving a system of algebraic equations thus greatly simplifying the problem. The proposed approach was based on the shifted Legendre tau and shifted Legendre collocation methods. In the limit, as approaches an integer value, the scheme provides solution for the integer-order differential equations. The fractional derivatives are described in the Caputo sense. A comparison between the proposed method and the Adomian Decomposition Method (ADM) is given. The study is conducted through illustrative example to demonstrate the validity and applicability of the presented method. The results reveal that the proposed method is very effective and simple. Moreover, only a small number of shifted Legendre polynomials is needed to obtain a satisfactory result.

Key words: Caputo's fractional derivative, operational matrix, Legendre polynomials, collocation method, Adomian decomposition method

INTRODUCTION

The number of scientific and engineering problems involving fractional calculus is already very large and still growing and perhaps the fractional calculus will be the calculus of twenty-first century. For example, fractional calculus is applied to model frequency dependent damping behavior of many viscoelastic materials [1], continuum and statistical mechanics, economics [2] and others [3-5]. Fractional differentials and fractional integrals provide more accurate models of systems under consideration. One of the most recent works on the subject of fractional calculus, i.e., the theory of derivatives and integrals of fractional (non-integer) order, is the book of Podlubny [6-8], which deals principally with fractional differential equations. And today, there are many works on fractional calculus [6, 9]. Most fractional differential equations do not have exact solutions, so approximation and numerical techniques must be used, such as Adomian's decomposition method [10-12], He's variational iteration method [9], homotopy perturbation method [13, 14], collocation method [15], Galerkin method and other methods [16-26].

Orthogonal functions have received considerable attention in dealing with various problems. The main characteristic behind the approach using this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. In this method, a truncated orthogonal series is used for numerical integration of differential equations, with the goal of obtaining efficient computational solutions. Several papers have appeared in the literature concerned with the application of shifted Legendre polynomials [27, 28]. In this paper we intend to extend the application of Legendre polynomials to solve fractional differential equations. Our main aim is to generalize Legendre operational matrix to fractional calculus. It is worthy to mention here that, the method based on using the operational matrix of an orthogonal function for solving differential equations is computer oriented.

The organization of this paper is as follows. In the next Section, Legendre operational matrix of the first derivative is obtained. Section 3 summarizes the Legendre operational matrix of the fractional derivative. In Section 4, the operational matrix of fractional derivative for nonlinear multi-order fractional differential equation is implemented. In Section 5, some numerical results are given to clarify the method. An implementation of the Adomian decomposition method is introduced in Section 6. Also a conclusion is given in Section 7. Note that we have computed the numerical results using Mathematica programming.

Corresponding Author: Dr. M.M. Khader, Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt
In this section some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development are given [8].

**Definition 1:** The Caputo fractional derivative operator $D^\alpha$ of order $\alpha$ is defined in the following form:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad m-1 < \alpha < m,$$

where $m-1 < \alpha < m$, $m \in \mathbb{N}$, $x > 0$.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^\alpha (f(x) \pm g(x)) = D^\alpha f(x) \pm D^\alpha g(x)$$

where $\lambda$ and $\mu$ are constants.

For the Caputo's derivative we have [8]:

$$D^\alpha C = 0, \quad C \text{ is a constant} \quad (1)$$

$$D^\alpha x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \alpha \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \alpha \end{cases} \quad (2)$$

We use the ceiling function $[\alpha]$ to denote the smallest integer greater than or equal to $\alpha$. Also $\mathbb{N}_0 = \{0, 1, 2, \ldots \}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties [6-8].

**LEGENDRE OPERATIONAL MATRIX OF THE FIRST DERIVATIVE**

The well known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula [29]:

$$L_{k+1}(z) = \frac{2k+1}{k+1} zL_k - \frac{k}{k+1} L_{k-1}(z), \quad k = 1, 2, \ldots$$

where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $x \in [0,1]$ we define the so called shifted Legendre polynomials by introducing the change of variable $z = 2x-1$. Let the shifted Legendre polynomials $L_k(2x-1)$ be denoted by $P_k(x)$. Then $P_k(x)$ can be obtained as follows:

$$P_{k+1}(x) = \frac{(2k+1)(2x-1)}{k+1} P_k(x) - \frac{k}{k+1} P_{k-1}(x), \quad k = 1, 2, \ldots \quad (3)$$

where $P_0(x)$ and $P_1(x) = 2x-1$. The analytic form of the shifted Legendre polynomials $P_k(x)$ of degree $k$ given by:

$$P_k(x) = \sum_{i=0}^k (-1)^{i+k} \frac{(k+i)!x^i}{(k-i)!} \quad (4)$$

Note that $P_k(0) = (-1)^k$ and $P_k(1) = 1$. The orthogonality condition is:

$$\int_0^1 P_i(x)P_j(x) dx = \begin{cases} \frac{1}{2i+1}, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases} \quad (5)$$

The function $u(x)$ square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as:

$$u(x) = \sum_{i=0}^\infty c_i P_i(x)$$

where the coefficients $c_i$ are given by:

$$c_i = \frac{1}{i!} \int_0^1 u(x)P_i(x) dx, \quad i = 1, 2, \ldots$$

In practice, only the first $(m+1)$-terms shifted Legendre polynomials are considered. Then we have:

$$u_m(x) = \sum_{i=0}^m c_i P_i(x) = C^T \Psi(x)$$

where the shifted Legendre coefficient vector $C$ and the shifted Legendre vector $\Psi(x)$ are given by:

$$C = [c_0, c_1, \ldots, c_m], \quad \Psi(x) = [P_0(x), P_1(x), \ldots, P_m(x)]^T \quad (7)$$

In the following theorem we will define the derivative of the vector $\Psi(x)$.

**Theorem 1:** The derivative of the vector $\Psi(x)$ can be expressed by:

$$\frac{d\Psi(x)}{dx} = \Delta(1)\Psi(x) \quad (8)$$
where $\Delta^{(1)}$ is the $(m+1) \times (m+1)$ operational matrix of derivative given by:

$$\Delta^{(1)}(d_i) = \begin{cases} 2(2j+1), & \text{for } j=i-k, \quad k=1,3,\ldots,m, \text{ if } m \text{ odd} \\ 0, & \text{if } m \text{ even} \end{cases}$$

For example for even $m$ we have:

$$\Delta^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 3 & 0 & \ldots & 0 \\ 1 & 0 & 5 & \ldots & 0 \\ 0 & 3 & 0 & 7 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 3 & 0 & 7 & \ldots & 0 \\ 1 & 0 & 5 & \ldots & 2m-3 & 0 \\ 0 & 3 & 0 & 7 & \ldots & 2m-1 & 0 \end{bmatrix}$$

In the following section we generalize the operational matrix of derivative of shifted Legendre polynomials given in (8) for fractional derivative.

**LEGENDRE OPERATIONAL MATRIX OF THE FRACTIONAL DERIVATIVE**

By using Eq.(8), for $n \in \mathbb{N}$, it is clear that:

$$\frac{d^n \Psi(x)}{dx^n} = (\Delta^{(1)})^n \Psi(x) = \Delta^{(n)} \Psi(x), \quad n=1,2,3,\ldots \tag{9}$$

**Theorem 2:** Let $\Psi(x)$ be shifted Legendre vector defined in (7) and also suppose $\alpha > 0$ then:

$$D^\alpha \Psi(x) \equiv \Delta^{(\alpha)} \Psi(x) \tag{10}$$

where $\Delta^{(\alpha)}$ is the $(m+1) \times (m+1)$ operational matrix of fractional derivative of order $\alpha$ in the Caputo sense and is defined as follows:

$$\Delta^{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \sum_{k=a}^{[\alpha]} w_{a,k} & \sum_{k=a}^{[\alpha]} w_{a,1,k} & \sum_{k=a}^{[\alpha]} w_{a,m,k} \\ \sum_{k=a}^{[\alpha]} w_{i,0,k} & \sum_{k=a}^{[\alpha]} w_{i,1,k} & \sum_{k=a}^{[\alpha]} w_{i,m,k} \\ \sum_{k=a}^{[\alpha]} w_{m,0,k} & \sum_{k=a}^{[\alpha]} w_{m,1,k} & \sum_{k=a}^{[\alpha]} w_{m,m,k} \end{bmatrix} \tag{11}$$

where $w_{i,j,k}$ is given by:

$$w_{i,j,k} = (2j+1) \sum_{\ell=0}^{[\alpha]} \frac{(-1)^{i+j+k+\ell}}{(i-k)! \ell!(\ell+j)! (k-a+1)! (j-\ell)!} \frac{(i+k)!}{(k+\ell-a+1)!} \tag{12}$$

Note that in $\Delta^{(\alpha)}$, the first $\lfloor \alpha \rfloor$ rows, are all zero [27].

**Proof:** Let $P_i(x)$ be a shifted Legendre polynomial then, by using (1), (2) and (4) we can find that:

$$D^\alpha P_i(x) = 0, \quad i = 0,1,2,\ldots, \lfloor \alpha \rfloor - 1, \quad \alpha > 0 \tag{13}$$

Also, by using (2) and (4) we can obtain (for $i = \lfloor \alpha \rfloor, \ldots, m$):

$$D^\alpha P_i(x) = \sum_{k=0}^{i} \frac{(-1)^{i+k}}{(i-k)! (k)!^2} D^\alpha x^k = \sum_{k=\lfloor \alpha \rfloor}^{i} \frac{(-1)^{i+k}}{(i-k)! (k)!} \frac{(i+k)!}{(k-\alpha)!} x^{k-\alpha} \tag{14}$$
Now, approximate $x^{a-k}$ by $m+1$-terms of shifted Legendre series, we have:

$$x^{a-k} = \sum_{j=0}^{m} d_{kj} P_{j}(x)$$

(15)

where

$$d_{kj} = \frac{1}{(2j+1)!} \sum_{\ell=0}^{j} \frac{(1)^{j+\ell}(\ell+j)!}{(j+\ell)!} \frac{1}{2^{j+\ell}} \frac{1}{(j+\ell)!} x^{j+\ell} dx = (2j+1) \sum_{\ell=0}^{j} \frac{(-1)^{j+\ell}(\ell+j)!}{(j+\ell)!} x^{j+\ell} dx = (2j+1) \sum_{\ell=0}^{j} \frac{(-1)^{j+\ell}(\ell+j)!}{(j+\ell)!} x^{j+\ell} dx$$

Employing Eqs. (14)-(16), we get:

$$D^{a} P_{i}(x) \equiv \sum_{k=0}^{i} \sum_{j=0}^{m} \frac{(-1)^{i+k} (i+k)!}{(i-k)!} w_{i,j,k} \sum_{j=0}^{m} \sum_{k=0}^{m} \frac{(-1)^{j+k} (j+k)!}{(j+k)!} g_{k-a+1} \frac{d_{j,k}}{d_{k,j}} P_{j}(x) = \sum_{k=0}^{m} \sum_{j=0}^{m} \frac{(-1)^{j+k} (j+k)!}{(j+k)!} w_{i,j,k} \sum_{j=0}^{m} \sum_{k=0}^{m} \frac{(-1)^{j+k} (j+k)!}{(j+k)!} g_{k-a+1} \frac{d_{j,k}}{d_{k,j}} P_{j}(x), \quad i = [a]...m$$

(17)

where $w_{i,j,k}$ is given in Eq. (12). Rewrite Eq. (17) as a vector form we have:

$$D^{a} P_{i}(x) \equiv \left[ \sum_{k=0}^{i} \frac{w_{i,k}}{a} \sum_{k=0}^{m} \frac{w_{i,k}}{a} \sum_{k=0}^{m} \frac{w_{i,k}}{a} \sum_{k=0}^{m} \frac{w_{i,k}}{a} \right] = \Psi_{i=a}...m$$

A combination of Eqs. (20) and (21) leads to the desired result.

**Remark:** If $\alpha = n \in \mathbb{N}$, then Theorem 2 gives the same result as Eq. (9).

In the following section, in order to show the high importance of operational matrix of fractional derivative, we apply it to solve the nonlinear multi-order fractional differential equation. The existence and uniqueness and continuous dependence of the solution to this problem are discussed in [30].

**OPERATIONAL MATRIX OF FRACTIONAL DERIVATIVE FOR NONLINEAR MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATION**

Consider the nonlinear multi-order fractional differential equation:

$$D^{a} u(x) = F(x,u(x),D^{b_{1}} u(x),...D^{b_{k}} u(x))$$

(20)

with initial conditions:

$$u^{(i)}(0) = d_{i}, \quad i = 0,1,...,n$$

(21)

where $n < \alpha \leq n+1, 0 < \beta_{1} < \beta_{2} < \ldots < \beta_{k} < \alpha$ and $D^{a}$ denotes the Caputo fractional derivative of order $\alpha$. It should be noted that $F$ can be nonlinear in general.

To solve problem (20)-(21) we approximate $u(x)$, $D^{a} u(x)$ and $D^{\beta_{i}} u(x)$, for $i = 1,2,...,k$ by the shifted Legendre polynomials as:

$$u(x) \equiv \sum_{i=0}^{m} c^{i} P_{i}(x) = C^{T} \Psi(x)$$

(22)

$$D^{a} u(x) \equiv C^{T} D^{a} \Psi(x) = C^{T} \Delta^{a} \Psi(x)$$

(23)

$$D^{\beta_{i}} u(x) \equiv C^{T} D^{\beta_{i}} \Psi(x) = C^{T} \Delta^{\beta_{i}} \Psi(x)$$

(24)

where vector $C = [c_{0}, c_{1},...,c_{m}]^{T}$ is an unknown vector.

By substituting these equations in Eq.(20) we get:

$$C^{T} \Delta^{(\alpha)} \Psi(x) \equiv F(x,C^{T} \Psi(x),C^{T} \Delta^{(\beta_{1})} \Psi(x),...,C^{T} \Delta^{(\beta_{k})} \Psi(x))$$

(25)

Also, by substituting Eqs. (22) in Eq. (21) we obtain:

$$u(0) = C^{T} \Psi(0) = d_{0}$$

$$u^{(i)}(0) = C^{T} \Delta^{(i)} \Psi(0) = d_{i}, \quad i = 1,2,...,n$$

In order to obtain the solution $u(x)$, we first collocate Eq. (25) at $m-n$ points. For suitable

collocation points we use the first m-n shifted Legendre roots of \( P_{m-n}(x) \). These equations together with Eq. (26) generate \( m+1 \) nonlinear equations which can be solved using Newton’s iterative method. Consequently \( u(x) \) given in Eq. (22) can be calculated.

**NUMERICAL IMPLEMENTATION**

Consider the following nonlinear differential equation:

\[
D^4 u(x) + D^2 u(x) + u^3(x) = x^9 \quad (27)
\]

subject to the initial conditions:

\[
u(0) = u^{(0)}(0) = u^{(0)}(0) = 0, \quad u^{(3)}(0) = 6 \quad (28)
\]

To solve the above problem, by applying the technique described in Section 4 with \( m = 4 \), we approximate solution as:

\[
u(x) = c_0 P_1(x) + c_1 P_2(x) + c_2 P_3(x) + c_3 P_4(x) = C^T \Psi(x)
\]

Here we have:

\[
\Delta^{(0)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Delta^{(1)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Delta^{(2)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Delta^{(3)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Delta^{(4)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1680 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Using Eq. (25) we have:

\[
C^T \Delta^{(4)} \Psi(x) + C^T \Delta^{(2)} \Psi(x) + (C^T \Psi(x))^3 = x^9
\]

Now we collocate Eq. (29) at the first root of \( P_5(x) \), i.e.,

\[
x_0 = \frac{1}{2} \sqrt{245-14\sqrt{70}} \approx 0.230765
\]

Also by using Eq. (26) we get:

\[
C^T \Psi(x) = c_0 - 6c_2 + 12c_3 - 20c_4 = 0
\]

\[
C^T \Delta^{(1)} \Psi(x) = 2c_2 - 6c_4 + 12c_5 - 20c_6 = 0
\]

\[
C^T \Delta^{(2)} \Psi(x) = 12c_4 - 60c_6 + 180c_7 = 0
\]

\[
C^T \Delta^{(3)} \Psi(x) = 120c_6 - 840c_8 = 0
\]

By solving Eqs. (29) and (30) we obtain:

\[
c_0 = \frac{1}{4}, \quad c_1 = \frac{9}{20}, \quad c_2 = \frac{1}{4}, \quad c_3 = \frac{1}{20}, \quad c_4 = 0
\]

Therefore

\[
u(x) = \left( \frac{1}{4}, \frac{9}{20}, \frac{1}{4}, \frac{1}{20}, 0 \right) \begin{pmatrix}
1 \\
2x - 1 \\
6x^2 - 6x + 1 \\
20x^3 - 30x^2 + 12x - 1 \\
70x^4 - 140x^3 + 90x^2 - 20x + 1
\end{pmatrix} = x^9
\]

which is the exact solution of the problem.
It is clear that in this example the present method can be considered as an efficient method.

**ADOMIAN DECOMPOSITION METHOD FOR NONLINEAR MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATION**

In order to obtain the solution of the problem (27)-(28) by means of ADM we apply the inverse operator on (27) and using the initial conditions (28), we can derive:

\[
\begin{align*}
    u(x) &= f(x) + L^{-1}_x\left(\frac{\partial}{\partial x} \right)^9 - L^{-1}_x [D^2 u(x) + u^3(x)] \\

    &= f(x) + \frac{2}{x^9} - L^{-1}_x [D^2 u(x) + u^3(x)] \\
\end{align*}
\]  

where \(\phi(x) = x^3\) is the solution of the homogenous differential equation \(D^4 u(x) = 0\) and the operator \(L^{-1}_x\) is an integral operator and given by

\[
L^{-1}_x(\cdot) = \int_{0}^{x} \int_{0}^{y} \int_{0}^{z} \int_{0}^{w} dxdydzdw .
\]

The ADM [31, 32] assumes that the unknown solution can be expressed by an infinite series of the form:

\[
u(x) = \sum_{n=0}^{\infty} u_n(x)
\]  

and the nonlinear operator term \(N(u) = u^3\) can be decomposed by an infinite series of polynomials given by:

\[
N(u) = \sum_{n=0}^{\infty} A_n
\]  

where the components \(u_n(x), n \geq 0\) will be determined recurrently and \(A_n\) are the so-called Adomian's polynomials of \(u_0, u_1, u_2, \ldots\) defined by:

\[
A_n = \frac{1}{n!} \frac{d^n}{dx^n} N\left( \sum_{i=0}^{n-1} \lambda_i u_i \right) \bigg|_{\lambda = 0}, \quad n \geq 0
\]  

these polynomials can be constructed for all nonlinearity according to algorithm set by Adomian. Substituting from (32), (33) into (31) and equating the similar terms in both sides of the equation, we get the following recurrence relation:

\[
\begin{align*}
    u_0(x) &= x^3 + L^{-1}_x(\frac{2}{x^9}) \\
    u_{n+1}(x) &= -L^{-1}_x (p^{7/2} u_n + A_n), \quad n \geq 0
\end{align*}
\]

where the first \(A_n\) Adomian's polynomials that represent the nonlinear term \(N(u) = u^3\) are given by:

\[
\begin{align*}
    A_0 &= u_0^3 \\
    A_1 &= 3u_0^2 u_1, \\
    A_2 &= 3(u_0^2 u_2 + u_1^2 u_0), \\
    A_3 &= u_1^3 + 6u_0^2 u_1 u_2 + 3u_0^2 u_3, \\
\end{align*}
\]

other polynomials can be generated in a like manner.

Now from the recurrence relationship (35) we obtain the following resulting components:

\[
\begin{align*}
    u_0(x) &= x^3 + 0.0000582751 x^{13} \\
    u_1(x) &= -0.0000582751 x^{13} - 0.0000157143 x^{27/2} - 8.22629 \times 10^{-10} x^{23} \\
    u_2(x) &= 0.0000157143 x^{27/2} + 4.1625 \times 10^{-6} x^{14} + 8.22629 \times 10^{-6} x^{23} + 3.71085 \times 10^{-6} x^{47/2} \\
    u_3(x) &= -4.1625 \times 10^{-6} x^{14} - 1.08375 \times 10^{-6} x^{29/2} - 3.71085 \times 10^{-10} x^{47/2} - 1.2432 \times 10^{-10} x^{24} \\
\end{align*}
\]

Having \(u_i(x), i = 0, 1, 2, \ldots n\), the solution is as follows:

\[
u(x) = \sum_{n=0}^{\infty} u_n(x)
\]  

The convergence of ADM is introduced in many papers, for example [33].

The behaviour of the exact solution \(u(x)\) and the approximate solution by means of ADM \(u_{ADM}(x) = \phi_3(x)\) is presented in the Fig. 1.
CONCLUSION

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason we need a reliable and efficient technique for the solution of fractional differential equations. This paper deals with the approximate solution of a class of multi-order fractional differential equations. The fractional derivatives are described in the Caputo sense. Our main aim is to generalize the Legendre operational matrix to the fractional calculus. In this approach, a truncated Legendre series together with the Legendre operational matrix of fractional derivatives are used for numerical integration of fractional differential equations. The main characteristic behind the approach using this technique is that it reduces such problems to those of solving a system of algebraic equations thus greatly simplifying the problem. The method is applied to solve nonlinear fractional differential equations. Illustrative example is included to demonstrate the validity and applicability of the presented technique. The comparison certifies that our method gives good results. All results obtained by using Mathematica version 6.

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