Analytic Approach to Investigation of Distributions of Stresses and Radial Displacement at the Thick-wall Cylinder under the Internal and External Pressures

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Abstract: In this letter, two analytical methods, Homotopy-Perturbation Method (HPM) and Adomian Decomposition Method (ADM), are introduced to obtain the approximate solution of the governing differential equation of thick-wall cylinder for radial displacement by applying the boundary condition. By comparing the results of two methods with the exact solution find that HPM yields to exact solution in one iteration and in ADM, solution is obtained after three iterations. He's Homotopy-Perturbation method which doesn't need small parameter is implemented for solving the differential equations and it is predicted that HPM can be founded widely applicable in engineering and in cases that don’t have exact solution this method can be used as semi-exact solution.

Keywords: Homotopy-perturbation method . adomian decomposition method . thick-wall cylinder . radial displacement . radial stress . circumferential stress

INTRODUCTION

In wide range of engineering problems and physics, encounter to solve the nonlinear differential equations [1-7]. We know that except a limited number of these problems, most of them don't have analytical solution. Therefore these nonlinear equations should be solved using other methods. Some of them are solved using numerical techniques and some are solved using the analytical method of perturbation. In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results. In the analytical perturbation method [8-11], we should exert the small parameter in the equation. Therefore, finding the small parameter and exerting it into the equation are difficulties of this method. Therefore, many effective methods are invented to eliminate the small parameter, such as homotopy perturbation method [12-19] and Adomian decomposition method [20-28].

Homotopy-perturbation method yields a very rapid convergence and usually, one iteration leads to high accuracy of solution. Adomian decomposition method considered the solution of functional equation as the sum of infinite series which converges rapidly to the accurate solutions [23-28]. In this paper we applied the homotopy perturbation and Adomian decomposition methods to solve the governing differential equation of thick-wall cylinder for radial displacement.

PROBLEM DESCRIPTION

Cylindrical coordinate system (r, θ, z): By writing the relations of stresses at cylindrical coordinate system related to thick-wall cylinder [29], the differential equation of motion (equations of equilibrium) are obtained as follows:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rr}}{r} - \frac{\sigma_{\theta \theta}}{r} + B_r = 0
\]

(1)

\[
\frac{\partial \sigma_{\theta \theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{2\sigma_{zz}}{r} + B_\theta = 0
\]

(2)

\[
\frac{\partial \sigma_{zz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{\sigma_{zz}}{r} + B_z = 0
\]

(3)

where \(\sigma_{rr}, \sigma_{\theta \theta}, \sigma_{zz}, \sigma_{\theta z} \) represent stress tensor component as shown in Fig. 1 and for radial symmetry \(B_r = 0, \sigma_{\theta \theta} = \sigma_{zz} = 0\).
For thick-wall cylinder that is under the internal and external pressures $\sigma_{zz} = 0$ and the stress tensor is obtained as follows:

$$\begin{bmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_{\theta \theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$ (4)

The equations of equilibrium for cylindrical coordinates (1)-(3) reduced to single equation:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_{zz}}{r} + B_1 = 0$$ (5)

**Strain-displacement relations and compatibility:** The strain-displacement relations for the thick-wall cylinder [30] is as follows:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}$$ (6)

$$\varepsilon_{\theta \theta} = \frac{u}{r}$$ (7)

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$ (8)

We Neglected body force components, Eq. (5) is written as follows:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_{zz}}{r} = 0$$ (9)

By applying the Hooke's relations, we will have:

$$\sigma_r = \frac{E}{1-\mu^2} (\varepsilon_{rr} + \mu \varepsilon_{\theta \theta})$$ (10)

$$\sigma_{zz} = \frac{E}{1-\mu^2} (\varepsilon_{zz} - \mu (\varepsilon_{rr} + \varepsilon_{\theta \theta}))$$ (11)

$$\sigma_n = E \varepsilon_n = -\mu (\varepsilon_{rr} + \varepsilon_{\theta \theta})$$ (12)

The $\mu$ is Poisson's ratio and $E$ is modulus of elasticity. At thick-wall cylinder $\sigma_{zz}$ is constant. By substituting the strain-displacement relations Eqs. (6) and (7) into Eqs. (10) and (11) we will have:

$$\sigma_r = \frac{E}{1-\mu^2} \left( \frac{\partial u}{\partial r} + \mu \frac{u}{r} \right)$$ (13)

$$\sigma_{zz} = \frac{E}{1-\mu^2} \left( \frac{\partial u}{\partial r} + \mu \frac{u}{r} \right)$$ (14)

where $u$ is radial displacement and can be expressed for $r (u = u(r))$.

By Substituting the Eqs.(13) and (14) at Eq. (9) yields the governing differential equation for the radial displacement:

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} = 0$$ (15)

At this paper is considered the powerful methods for solving the governing deferential equation and found $u (r)$.

**BASIC IDEA OF HOMOTOPY PERTURBATION METHOD**

To illustrate the basic idea of HPM [12-19], we consider the following nonlinear differential equation [33]:

$$A(u) - f(r) = 0, r \in \Omega$$ (16)

With the boundary conditions of:

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma$$ (17)

where $A$ is a general differential operator, $B$ a boundary operator, $f(r)$ a known analytical function and $\Gamma$ is the boundary of the domain $\Omega$. The operator $A$ can be divided into a linear part $L$ and nonlinear part $N(u)$. Eq. (16) can be rewritten as:

$$L(u) + N(u) = f(r) = 0$$ (18)

We construct a homotopy $v (r,p): \Omega \times [0,1] \rightarrow \mathbb{R}$, which satisfies:

$$L(u) + p N(u) = f(r) \quad 0 \leq p \leq 1$$ (19)

$$v(r,0) = u_0 \quad \text{subject to boundary conditions}$$ (20)
\( H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)], p \in [0,1], r \in \Omega \) \hspace{1cm} (19)

For \( p = 0 \) and \( p = 1 \) Eq. (19) is reduced to following equations:

\[ H(v,0) = L(v) - L(u_0) = 0 \] \hspace{1cm} (20)

\[ H(v,1) = A(v) - f(r) = 0 \] \hspace{1cm} (21)

The changing process of \( p \) from zero to unity is just that of \( \nu(r,p) \) from \( u_0(r) \) to \( u(r) \).

We consider \( \nu \) as follows:

\[ \nu = \zeta + pv_1 + \cdots \] \hspace{1cm} (22)

Setting \( p = 1 \) yields in approximate solution of Eq. (16) to:

\[ u = \lim_{p \to 1} \nu = v_0 + v_1 + \cdots \] \hspace{1cm} (23)

The above convergence is discussed in [31, 32].

**Implementation of HPM:** By applying the HPM to Eq. (15), the differential equation of thick-wall cylinder for the radial displacement is separated to the linear and nonlinear parts. we will have:

\[ L(v) = \frac{d^2v(r)}{dr^2} + \frac{1}{r} \frac{dv(r)}{dr} - \frac{v(r)}{r^2} \] \hspace{1cm} (24)

\( L(u_0) \) is approximation of initial conditions and is defined:

\[ L(u_0) = \frac{d^2u_0(r)}{dr^2} + \frac{1}{r} \frac{du_0(r)}{dr} - \frac{u_0(r)}{r^2} \] \hspace{1cm} (25)

We applied \( v \) as follows [34]:

\[ v = v_0 + pv_1 + p^2v_2 \] \hspace{1cm} (26)

Therefore, Eqs. (24-26) are substituted at Eq. (19) (the nonlinear term \( N(v) \) is zero.)

\[ \frac{d^2[v_0 + pv_1 + p^2v_2]}{dr^2} + \frac{1}{r} \frac{dv_0 + pv_1 + p^2v_2}{dr} - \frac{v_0 + pv_1 + p^2v_2}{r^2} \]

\[ + (p-1)\left[ \frac{d^2u_0(r)}{dr^2} + \frac{1}{r} \frac{du_0(r)}{dr} - \frac{u_0(r)}{r^2} \right] = 0 \] \hspace{1cm} (27)

By rearranging the powers of \( p \) we will have:

\[ p^0 : \frac{d^2v_0(r)}{dr^2} + \frac{1}{r} \frac{dv_0(r)}{dr} - \frac{v_0(r)}{r^2} \]

\[ - \left[ \frac{d^2u_0(r)}{dr^2} + \frac{1}{r} \frac{du_0(r)}{dr} - \frac{u_0(r)}{r^2} \right] = 0 \] \hspace{1cm} (28)

\[ p^1 : \frac{d^2v_1(r)}{dr^2} + \frac{1}{r} \frac{dv_1(r)}{dr} - \frac{v_1(r)}{r^2} \]

\[ - \left[ \frac{d^2u_1(r)}{dr^2} + \frac{1}{r} \frac{du_1(r)}{dr} - \frac{u_1(r)}{r^2} \right] = 0 \] \hspace{1cm} (29)

\[ p^2 : \frac{d^2v_2(r)}{dr^2} + \frac{1}{r} \frac{dv_2(r)}{dr} - \frac{v_2(r)}{r^2} = 0 \] \hspace{1cm} (30)

By solving above equations, \( v \) is determined. From solving Eq. (28), we will determine:

\[ v_0(r) = u_0(r) = \zeta r + c_1 \frac{1}{r} \] \hspace{1cm} (31)

Similarly for other equations:

\[ v_1(r) = c_2 r + c_3 \frac{1}{r} \] \hspace{1cm} (32)

And similarly:

\[ v_2(r) = c_4 r + c_5 \frac{1}{r} \] \hspace{1cm} (33)

According to definition of HPM, by substituting the Eqs.(31-33) in Eq. (26) gives the equation for \( v \) and as \( p \to 1 \), then \( v \to u \). On the other hand, it is the answer for the governing equation of thick-wall cylinder Eq. (15). the exact solution of \( u(r) \) is:

\[ u(r) = \zeta r + c_6 \frac{1}{r} \] \hspace{1cm} (34)

**THEORY OF ADOMIAN DECOMPOSITION METHOD**

The principal algorithm of the Adomian decomposition method when applied to a general nonlinear equation is in the form [20-28]:

\[ Lu + Ru + Nu = g \] \hspace{1cm} (35)

The linear terms are decomposed into \( L+R \), while the nonlinear terms are represented by \( Nu \). \( L \) is taken as the highest-order derivative to avoid difficult integration involving complicated green’s functions and \( R \) is remainder of the linear operator. \( L^{-1} \) was regarded as the inverse operator of \( L \) and is defined by a definite integration from 0 to \( x \).

\[ [L^+] (x) = \int_0^x f(v) dv \]  

(36)

If \( L \) is a second-order operator, \( L^+ \) is a twofold indefinite integral.

\[ L^+L u = u - u(0) - \frac{\delta u(0)}{\delta x} \]  

(37)

Operating on both sides of Eq. (35) With \( L^{-1} \) yields:

\[ L^+Lu = L^+g - L^+(Ru) - L^+(Nu) \]  

(38)

Comparing Eqs. (44) and (45) gives:

\[ u = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \]  

(39)

Where:

\[ f(x) = f_1(x) + f_2(x) \]  

(40)

\( f(x) \) is decomposed to two parts: \( f_1(x) \) and \( f_2(x) \). \( f_1(x) \) is the part corresponding to the source term \( g \) in original differential equation and \( f_2(x) \) arises from the prescribed initial or solution \( u(x) \). The decomposition technique represented the solution of Eq. (39) as series:

\[ u(x) = \sum_{n=0}^{\infty} u_n \]  

(41)

The nonlinear operator, \( Nu \) is decomposed by an infinite series of polynomials as:

\[ Nu = \sum_{n=0}^{\infty} A_n (u_0, u_1, \ldots, u_n) \]  

(42)

where \( A_n \) is Adomian’s polynomial of \( u_0, u_1, \ldots, u_n \) and are obtained from the formula:

\[ A_n = \frac{1}{n!} \frac{d^n}{dx^n} N(\sum_{i=0}^{n} \lambda^i u_i) \]  

(43)

By using recursive relation it can be written as:

\[ u_n(x) = f(x) \]  

\[ u_{n+1}(x) = -L^{-1}(Ru_n) - L^{-1}(Nu_n), n \geq 0 \]  

(44)

This leads to the solution in a series form. The solution \( u_n (x,t) \) in a closed form is readily obtained [34, 35].

**Application of ADM:** We applied the ADM for the governing equation of thick-wall cylinder and by using the initial condition as follows [34]:

\[ u_n = f(r) \text{That} u_0 = c_1 + c_2 \frac{1}{r} \]  

(45)

To solve Eq. (15) we must find the \( f_1(x) \) and \( f_2(x) \).

\( f_1(x) \) is arisen from integrating g-term. It is expressed by Eq. (46) and \( f_2(x) \) is arisen from the prescribed initial or boundary conditions. We will have:

\[ f_1(r) = L^{-1}_n(g(r)) = c_0 r \]  

(46)

The operator \( L^{-1}_n \) is twofold integral.

\[ f_2(r) = c_1 + c_2 \frac{1}{r} \]  

(47)

Therefore,

\[ f(r) = f_1(r) + f_2(r) \]  

(48)

According to the definitions of the ADM at previous section

\[ Lu = \frac{d^2 u}{dr^2} \]  

is second-order derivative and

\[ Ru = \frac{1}{r} \frac{du}{dr} \frac{1}{r^2} u(r) \]

and \( Nu \) is zero.

By applying the recursive relations, Eq. (44), we will have:

\[ u_0(r) = -L^{-1}_n \left( \frac{1}{r} \frac{du}{dr} \frac{1}{r^2} u_0(r) \right) = c_1 \]  

(49)

\[ u_1(r) = -L^{-1}_n \left( \frac{1}{r} \frac{du}{dr} \frac{1}{r^2} u_1(r) \right) = c_2 \]  

(50)

\[ u_2(r) = -L^{-1}_n \left( \frac{1}{r} \frac{du}{dr} \frac{1}{r^2} u_2(r) \right) = c_3 \]  

(51)

After third iteration, we will have:

\[ u(r) = \sum_{n=0}^{\infty} u_n(r) = u_0 + u_1 + u_2 + u_3 = c_0 + c_1 \frac{1}{r} \]  

(52)
COMPARING THE RESULTS

Finally, we completed the solution by determining the constants ($c_7$ and $c_8$) by using boundary conditions is as follows [30]:

$$\begin{align*}
\text{at} r = a, \sigma_r &= -p_1 \\
\text{at} r = b, \sigma_r &= -p_2
\end{align*} \quad (53)$$

$a$ and $b$ are sequentially inner, outer radiuses of cylinder (m). Substitution of $u(r)$ in (13-14) gives stress relations as follows:

$$\begin{align*}
\sigma_r &= \frac{E}{1-\mu^2} \left[ (1+\mu)c_7 - (1-\mu) \frac{c_8}{r} \right] \quad (54) \\
\sigma_{\theta\theta} &= \frac{E}{1-\mu^2} \left[ (1+\mu)c_7 - (1-\mu) \frac{c_8}{r^2} \right] \quad (55)
\end{align*}$$

We applied boundary conditions Eq. (53) into Eqs. (54) and (55) so:

$$\begin{align*}
-p_1 &= \frac{E}{1-\mu^2} \left[ (1+\mu)c_7 - (1-\mu) \frac{c_8}{a} \right] \\
-p_2 &= \frac{E}{1-\mu^2} \left[ (1+\mu)c_7 - (1-\mu) \frac{c_8}{b^2} \right]
\end{align*}$$

By solving two equations, we will have:

$$\begin{align*}
c_7 &= -\frac{1-\mu}{E} \left( \frac{p_1 b^2 - p_2 a^2}{b^2 - a^2} \right) \\
c_8 &= -\frac{1+\mu}{E} \left( \frac{a \sqrt{b^2 - a^2}}{b^2 - a^2} \right) \quad (56, 57)
\end{align*}$$

We considered the thick-wall cylinder with two specific states of boundary conditions:

a) Only external pressure ($p_1 = 0$) [30]:

$$\sigma_r = -\frac{p_1 b^2}{b^2 - a^2} \left( \frac{a^2}{r^2} \right) \quad (58)$$

$$\sigma_{\theta\theta} = -\frac{p_1 b^2}{b^2 - a^2} \left( \frac{a^2}{r^2} \right) \quad (59)$$

$$u(r) = -\frac{p_1 b^2 r}{E(b^2 - a^2)} \left( \frac{(1-\mu) + (1+\mu) \frac{a^2}{r^2}}{r} \right)$$

The exact solution of Eq. (15) and distribution of stresses and radial displacement is discussed in References [29, 30]. The graphic results of HPM and ADM is shown in Fig. 1-3. For the thick-wall cylinder under the internal pressure ($p_2$). We supposed that the thick-wall cylinder is made of steel with properties $E = 200$Gpa, $v = 0.29$.

Fig. 2: Elastic radial stress of thick-wall cylinder with internal pressure ($p_1$), $a=10\text{mm}$, $b=50\text{mm}$, $p_1=150\text{Mpa}$.

Fig. 3: Elastic circumferential stress of thick-wall cylinder with internal pressure ($p_1$), $a=10\text{mm}$, $b=50\text{mm}$, $p_1=150\text{Mpa}$.
The radial stress for HPM and ADM results is compared in Fig. 2. The maximum absolute value of the radial stress is at inner surface \( r = a \).

The circumferential stresses are compared for two methods in Fig. 3. The maximum value of is circumferential stress occurred at the \( r = a \).

The radial displacement is compared in Fig. 4. Maximum value is at the \( r = a \).

In Fig. 5-7, the graphic results are shown radial stress for the thick-wall cylinder under the external pressure \( (p_2) \). The radial stress is shown in Fig. 5, the maximum absolute value of \( \sigma_r \) is occurred at the \( r = b \).

The circumferential stress is shown in Fig. 6 and the maximum absolute value of it is occurred at the \( r = a \).

The radial displacement is shown in Fig. 7, the maximum of \( u(r) \) is occurred at the \( r = b \).

As it can be seen the results of applying HPM and ADM is the same. The homotopy-perturbation method is extremely simple, easy to apply and gives a very good accuracy and needs less computation [34-41].

**CONCLUSIONS**

In this paper, the homotopy perturbation and Adomian decomposition method are applied to solve the differential equation of thick-wall cylinder for radial displacement. Comparison of HPM and ADM show that although the results of these methods are same but HPM yields to exact solution in one iteration without complicated but ADM is difficult at construction functional using Lagrange multipliers and the results are obtained after three iteration. The rate of convergence in HPM is high and the algorithm of solution is simple, rapid with high accuracy. Finally,
it has been attempted to show the capabilities and wide-range applications of the homotopy perturbation method

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