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A Note on the Chaudhry and Zubair (2002)'s Generalization of the Generalized Inverse Gaussian Distribution – Review of the Model, Distributional Properties and Applications

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Abstract: The objective of this paper is a statistical analysis of the Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution [cf. M. A. Chaudhry and S. M. Zubair [1], On A Class of Incomplete Gamma Functions, with Applications, Eq., p. 195] and to draw some inferences on it. The model has been reviewed first. Then, the statistical analysis of the model has been investigated. For this, several new distributional properties of the distribution have been derived, including the reliability analysis, the estimation of the parameters and computations of percentage points. We have used some real life-time data to show the applications of the Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution.

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Key words: Generalizations of Generalized Inverse Gaussian Distribution · Estimation · Incomplete Gamma Functions · Reliability

INTRODUCTION

The generalized inverse Gaussian (GIG) distribution has received special attention in view of its wide applications in many areas of research such as actuaries, biomedicine, demography, environmental and ecological sciences, finance, lifetime data, reliability theory and traffic data, among others. It was first proposed by Halphen in 1946; for details, see Perrault *et al*. [2] and Seshadri [3]. Later on, it was studied by many authors and researchers, such as Good [4], Tweedie [5, 6], Sichel [7], Folks and Chhikara [8], Wise [9], Barndorff-Nielsen [10], Jorgensen [11], Iyengar and Liao [12], Seshadri and Wesolowski [13], Wesolowski [14], Chaudhry and Zubair [1, 15], Chou and Huang [16], Al-Saqabi *et al*. [17] and Lemonte and Cordeiro [18], among others. For detailed and extensive discussions on the GIG distributions, the interested readers are also referred to Johnson *et al*. [19], Chhikara [20], Chhikara and Folks, [21], Seshadri [22, 23] and Marshall and Olkin [24]. For the computations of the percentage points of the inverse Gaussian distribution, see Kallioras and Koutrouvelis [25] and Koziol [26]. For extensive tables for the order statistics and the related calculations of moments of the inverse Gaussian distribution, please refer to Balakrishnan and Chen [27]. Recently, Chaudhry and Zubair ([1], p. 195) introduced a generalization of the generalized inverse Gaussian distribution (GGIGD). However, to the best knowledge of the authors, no attempts have been made to the statistical analysis of the Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution (GGIGD) since it appeared in their seminal work, viz.: Chaudhry and Zubair [1]. Motivated by these facts, we review the model first. Then, the statistical analysis of the model has been investigated and some inferences are drawn on it. We believe that the findings of this paper will be quite useful for the researchers and practitioners in various fields of theoretical and applied sciences, such as biomedicine, demography, environmental science, ecological science, finance, lifetime and quantum plasmadynamics, among others.

The organization of the paper is as follows: In Section 2, we give a description the Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution (GGIGD) model. Several new distributional properties, including the reliability analysis, moments and Shannon entropy are given in Section 3. The estimation of the parameters and computations of percentage points are provided in Sections 4 and 5, respectively. We have used some real life-time data to show the applications of the Chaudhry and Zubair (2002)'s distribution in Section 6. We have provided the concluding remarks in section 7.

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Chaudhry and Zubair's Generalization of the Generalized Inverse Gaussian Distribution (GGIGD) Model: In what follows, we will first provide the definition of the Chaudhry and Zubair [1]'s GGIGD and then will derive its several special cases and distributional properties.

Probability Density Function of the Chaudhry and Zubair [1]'s GGIGD: For a continuous positive random variable X, Chaudhry and Zubair ([1], p. 195) introduced a generalization of the generalized inverse Gaussian distribution with its probability density function (pdf) given by

$$f_X(x) = C\, x^{\alpha - 1} \exp\left(-a\,x\right) W_{0,\, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right), \tag{1}$$

where $x > 0$, $\nu \geq 0$, $a > 0$, $b \geq 0$, $-\infty < \alpha < \infty$, and

$$C = C\left(\alpha;\, a,\, b,\, \nu\right) = \left(\int_0^{\infty} x^{\alpha - 1} \exp\left(-a\,x\right) W_{0,\, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right) dx\right)^{-1} \tag{2}$$

Denotes the normalizing constant and $W_{k,\nu}(z)$ denotes the Whittaker function for reals κ and ν and real argument z; see, for example, Lebedev [28].

Using the Eq. 4.14, P. 197 of Chaudhry and Zubair [1], the normalizing constant is easily given by the following formula.

$$C\,(\alpha;\, a,\, b,\, \nu)$$

$$= \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} \left(ab\right)^{1/2} G_{0,\,4}^{4,\,0}\left(\frac{\left(ab\right)^2}{16}\middle| \frac{1}{2}\left(\nu + \frac{1}{2}\right), \frac{-1}{2}\left(\nu + \frac{1}{2}\right), \frac{1}{2}\left(\alpha + \frac{1}{2}\right), \frac{1}{2}\left(\alpha - \frac{1}{2}\right)\right)\right)^{-1} \tag{3}$$

where $G_{0,4}^{4,0}(.)$ denotes the Meijer G-function; see, for example, Mathai [29].

Remark 2.1 (Chaudhry and Zubair's GGIGD in Terms of the Confluent Hypergeometric Function): It is known that the Whittaker function, $W_{\lambda,\nu}(z)$ is related to the confluent hypergeometric function $\Psi([\ ], [\ ]; z)$ by the following formula

$$W_{\lambda,\, \mu}\left(z\right) = z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}} \Psi\left(\frac{1}{2} - \lambda + \mu,\, 2\mu + 1,\, z\right), \arg\left|z\right| < \pi,$$

Lebedev ([28], Eq. 9.13.16, p. 274), hence, using the above-mentioned formula, the equations (1) and (2) can easily be transformed in terms of the Whittaker function. Thus, we have;

$$f_X(x) = C\left(2b\right)^{\nu + 1} x^{\alpha - \nu - 2} e^{-\left(ax + \frac{b}{x}\right)} \Psi\left(\nu + 1,\, 2\left(\nu + 1\right);\, \frac{2b}{x}\right), \tag{4}$$

and

$$C = C\left(\alpha;\, a,\, b,\, \nu\right) = \left(\left(2b\right)^{\nu + 1} \int_0^{\infty} x^{\alpha - \nu - 2} e^{-\left(ax + \frac{b}{x}\right)} \Psi\left(\nu + 1,\, 2\left(\nu + 1\right);\, \frac{2b}{x}\right) dx\right)^{-1}, \tag{5}$$

where $x > 0$, $\nu \geq 0$, $a > 0$, $b \geq 0$, $-\infty < \alpha < \infty$.

Remark 2.2 (Chaudhry and Zubair's GGIGD in Terms of the Macdonald Function): Since it is known from Gradshteyn and Ryzhik ([30], Eq. 9.235.2, p. 1062) that that the Whittaker function, $W_{0,\mu}(z)$ is related to the Macdonald function $K_\mu(z)$ by the following formulas.

$$K_\mu\left(z\right) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,\, \mu}\left(2z\right),$$

that is,

$$W_{0,\, \mu}\left(2z\right) = \left(\frac{2z}{\pi}\right)^{1/2} K_\mu\left(z\right),$$

Hence, in view of above-mentioned formulas, the equations (1) and (2) can also be easily simplified in terms of the Macdonald function. Thus, we have.

$$f_X(x) = C\left(\frac{2b}{\pi}\right)^{\frac{1}{2}} x^{\alpha-\frac{3}{2}} e^{-ax} K_{v+\frac{1}{2}}\left(\frac{b}{x}\right), \tag{6}$$

and

$$C = C(\alpha; a, b, v) = \left(\left(\frac{2b}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \left(\frac{2b}{\pi}\right)^{\frac{1}{2}} x^{\alpha-\frac{3}{2}} e^{-ax} K_{v+\frac{1}{2}}\left(\frac{b}{x}\right) dx\right)^{-1}, \tag{7}$$

where $x > 0$, $v \geq 0$, $a > 0$, $b \geq 0$, $-\infty < \alpha < \infty$.

Remark 2.3 (Special Cases): Here, based on our key parameter $v \geq 0$, by taking some of its values, we will derive some special cases of the Chaudhry and Zubair (2002)'s GGIGD. These are given below:

Case (I) (when $v = 0$): Taking $v = 0$ in the Eq. (6), we have,

$$f_X(x) = C\left(\frac{2b}{\pi}\right)^{\frac{1}{2}} x^{\alpha-\frac{3}{2}} e^{-ax} K_{\frac{1}{2}}\left(\frac{b}{x}\right). \tag{8}$$

But, according to Melrose ([31], Eq. 2.4.37, P. 68), the Macdonald function of half-integer order can be expressed in terms of a rational function times an exponential function, that is, that $K_{\frac{1}{2}}(t) = \left(\frac{\pi}{2t}\right)^{\frac{1}{2}} e^{-t}$. Thus, using this formula in the Eq. (8) and after simplifying, we have;

$$f_X(x) = C x^{\alpha-1} \mathrm{e}^{-\frac{b}{x}-ax}. \tag{9}$$

Integrating the Eq. (9) from $x = 0$ to $x = \infty$ and using Gradshteyn and Ryzhik ([30], Eq. 3.471.9, p. 340), we have;

$$C = \frac{1}{2}\left(\frac{a}{b}\right)^{\frac{\alpha}{2}} \frac{1}{K_\alpha\left(2\sqrt{ab}\right)},$$

and, thus, Eq. (1.9) becomes,

$$f_X(x) = \frac{1}{2}\left(\frac{a}{b}\right)^{\frac{\alpha}{2}} \frac{1}{K_\alpha\left(2\sqrt{ab}\right)} x^{\alpha-1} \mathrm{e}^{-\frac{b}{x}-ax}, x > 0, a > 0, b > 0, -\infty < \alpha < \infty, \tag{10}$$

which is the pdf of the generalized inverse Gaussian distribution (GIG); see, Good [4] and Jorgensen [19].

Case (II) (when $v = 0$ and $\alpha = -\frac{1}{2}$): Taking $\alpha = -\frac{1}{2}$ and re-parametrization of the parameters a and b in the Eq. (10), such that $a = \frac{\lambda}{2\mu^2}$ and $b = \frac{\lambda}{2}$, it is easily seen that the Chaudhry and Zubair [1]'s GGIGD, with the pdf (1), easily reduces to the well-known two-parameter inverse Gaussian distribution, with the pdf given by;

$$f_X(x) = \sqrt{\frac{\lambda}{2\pi}} x^{-\frac{3}{2}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), x > 0, \lambda > 0, \mu > 0,$$

Case (III) (Three-parameter Inverse Gaussian Distribution): As pointed outed out by Balakrishnan and Chen ([27], p.6), "from the standard two-parameter form of the inverse Gaussian distribution, as mentioned in Case II above, a three-parameter inverse Gaussian distribution can be obtained easily by introducing a threshold (location) parameter, γ; see Cheng and Amin [32]." Thus, by the re-parametrization of the parameters a and b in the Eq. (10), such that $a = \frac{\lambda}{2\mu^2}$ and $b = \frac{\lambda}{2}$, its pdf is given as follows:

$$f_X(x) = \sqrt{\frac{\lambda}{2\pi(x-\gamma)^3}} \exp\left(-\frac{\lambda((x-\gamma)-\mu)^2}{2\mu^2(x-\gamma)}\right),$$

where $\lambda(>0)$: scale parameter; γ (real): location parameter; $\mu(>0)$: location parameter; and $\gamma + \mu$ is the mean, such that $\gamma < x < +\infty$.

Case (IV) (when $v = 1$): Taking $v = 1$ in the Eq. (6), we have.

$$f_X(x) = C\left(\frac{2b}{\pi}\right)^{\frac{1}{2}} x^{\alpha - \frac{3}{2}} e^{-ax} K_{\frac{3}{2}}\left(\frac{b}{x}\right), \tag{11}$$

Again, as shown by Melrose ([31], Eq. 2.4.37, P. 68), we have $K_{\frac{3}{2}}(t) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} t^{-\frac{3}{2}}(1+t)e^{-t}$. Thus, using this formula in the Eq. (11) and after simplifying, we have

$$f_X(x) = C b^{-1} x^{\alpha} e^{-\frac{b}{x} - ax}\left(1 + \frac{b}{x}\right). \tag{12}$$

Integrating the Eq. (12) from $x = 0$ to $x = \infty$, using Gradshteyn and Ryzhik ([30], Eq. 3.471.9, p. 340) and simplifying, we obtain the following formula for the normalizing constant:

$$C = \left(2\left(\frac{b}{a}\right)^{\frac{\alpha}{2}}\left\{\left(\frac{1}{ab}\right)^{\frac{1}{2}} K_{\alpha+1}\left(2\sqrt{ab}\right) + K_{\alpha}\left(2\sqrt{ab}\right)\right\}\right)^{-1}. \tag{13}$$

Case (V) (when $v = 2$): Taking $v = 2$ in the Eq. (6), we have

$$f_X(x) = C\left(\frac{2b}{\pi}\right)^{\frac{1}{2}} x^{\alpha - \frac{3}{2}} e^{-ax} K_{\frac{5}{2}}\left(\frac{b}{x}\right). \tag{14}$$

According to Melrose ([31], Eq. 2.4.37, P. 68), we have $K_{\frac{5}{2}}(t) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} t^{-\frac{5}{2}}\left(3 + 3t + t^2\right)e^{-t}$. Thus, using this formula in the Eq. (14) and after simplifying, we have;

$$f_X(x) = C b^{-2} x^{\alpha+1} e^{-\frac{b}{x} - ax}\left(3 + \frac{3b}{x} + \frac{b^2}{x^2}\right). \tag{15}$$

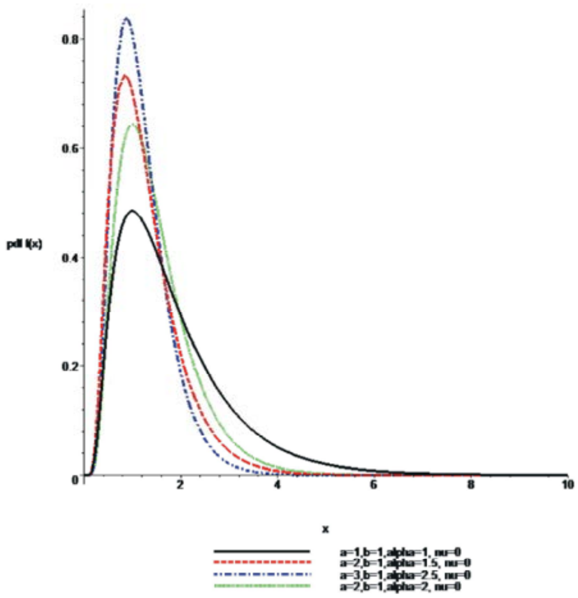
Integrating the Eq. (15) from $x = 0$ to $x = \infty$, using Gradshteyn and Ryzhik ([30], Eq. 3.471.9, p. 340) and simplifying, we obtain the following formula for the normalizing constant:

$$C = \left(2\left(\frac{b}{a}\right)^{\frac{\alpha}{2}}\left\{\frac{3}{ab} K_{\alpha+2}\left(2\sqrt{ab}\right) + 3\left(\frac{1}{ab}\right)^{\frac{1}{2}} K_{\alpha+1}\left(2\sqrt{ab}\right) + K_{\alpha}\left(2\sqrt{ab}\right)\right\}\right)^{-1}. \tag{16}$$

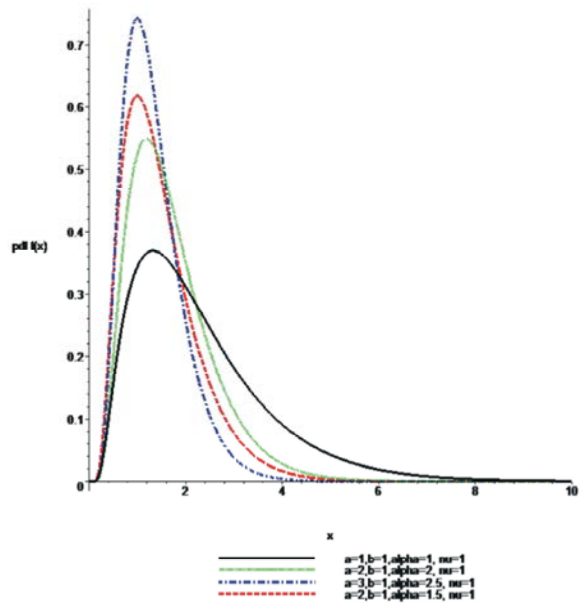
Remark 2.4 (Other Distributional Relationships): As pointed out by Chaudhry and Zubair [1], it can easily be seen that, by a simple transformation of the variable X or by taking special values of the parameters $\{\alpha, a, b, v\}$ in equation (1), a number of distributions, such as Weibull, gamma, inverse gamma, the hyperbolic, the inverse Gaussian, the generalized inverse Gaussian, Erlang, exponential, Rayleigh and chi-square, are special cases of (1). Furthermore, according to Chaudhry and Zubair [1], "the study of the probability model (1) will provide a unified approach to the systematic analysis of the probability densities encountered in forestry, reliability theory and in demographic rates".

Possible Shapes of the Chaudhry and Zubair [1]'s GGIGD pdf: The possible shapes of the pdf (1) are given for some selected values of the parameters in Figure 1 (a – c) below. The effects of the parameters can easily be seen from these graphs. For example, it is clear from these plots that the distributions of the GGIGD are positively right skewed with longer and heavier right tails for selected values of the parameters.

(a) $\{\alpha$ 1,1.5, 2, 2.5; a = 1,2,3; b = 1; v = 0$\}$



(b) $\{\alpha$ = 1,1.5, 2,2.5; a = 1,2,3; b = 1; v = 1$\}$



(c) $\{\alpha$ = 1,1.5, 2,2.5; a = 1,2,3; b = 1; v = 0$\}$

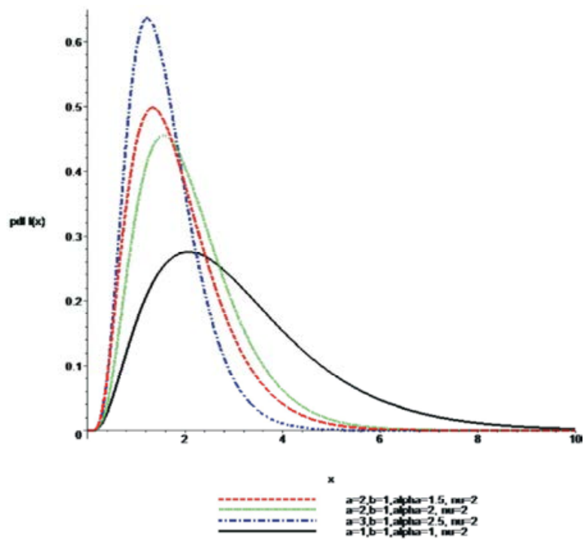


Fig. 1 (a – c): Plots of the Chaudhry and Zubair [1]'s GGIGD pdf (1)

Some Distributional Properties of the Chaudhry and Zubair [1]'s GGIGD Model: In what follows, we provide some distributional properties the Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution.

Cumulative Distribution Function: The cumulative distribution function (cdf) corresponding to the pdf (1) is given by;

$$F_X(x) = C\int_0^x t^{\alpha-1}\exp(-at)W_{0,\nu+\frac{1}{2}}\left(\frac{2b}{t}\right)dt, \tag{17}$$

which obviously cannot be integrated analytically in closed form and so should be evaluated numerically.

Furthermore, since it is known from Gradshteyn and Ryzhik ([30], Eq. 9.235.2, P. 1062) that $W_{0,\nu}(2z) = \left(\frac{2z}{\pi}\right)^{1/2} K_\nu(z)$, hence, in view of this formula, the equations (17) can be simplified in terms of the Macdonald function $K_\nu(z)$.

Remark 3.1: Cumulative Distribution Function in terms of the Generalizations of the Generalized Incomplete Gamma Functions: Chaudhry and Zubair [15, 1] introduced the following generalizations of the generalized incomplete gamma functions:

$$\gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2}\int_0^x t^{\alpha-\frac{3}{2}}\exp(-t)K_{\nu+\frac{1}{2}}\left(\frac{b}{t}\right)dt, \tag{18}$$

and

$$\Gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2}\int_x^\infty t^{\alpha-\frac{3}{2}}\exp(-t)K_{\nu+\frac{1}{2}}\left(\frac{b}{t}\right)dt, \tag{19}$$

where α, x are complex parameters, b is a complex variable and $K_\nu(z)$ denotes the modified Bessel function of the second kind or the Macdonald function for complex parameter ν and complex argument z. Note that for ν real and z positive, $K_\nu(z)$ is real. For details on the theory and analytical properties of Bessel functions, the interested readers are referred to Watson [33]. Thus, by using the definitions (18) and (19), the cumulative distribution function (cdf), $F_X(x)$, the reliability function, $R_X(x)$ and the hazard functions, $h(x)$, corresponding to the pdf (1), are respectively given in terms of the generalizations of the generalized incomplete gamma functions by;

$$F_X(x) = C a^{-\alpha}\gamma_\nu(\alpha, ax; ab), \tag{20}$$

$$R(x) = C a^{-\alpha}\Gamma_\nu(\alpha, ax; ab), \tag{21}$$

and

$$h(x) = \frac{a^\alpha \exp(-ax)W_{0,\nu+\frac{1}{2}}\left(\frac{2b}{x}\right)}{\Gamma_\nu(\alpha, ax; ab)}. \tag{22}$$

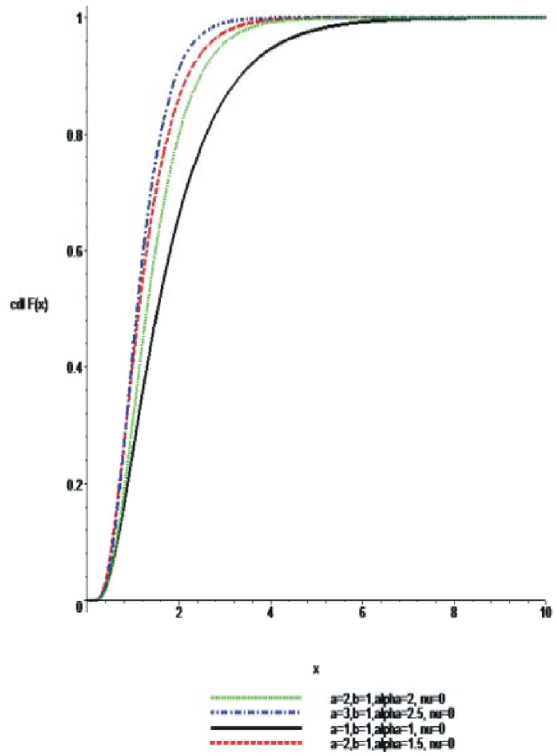
Possible Shapes of the Cumulative Distribution Function of the Chaudhry and Zubair [1]'s GGIGD: The possible shapes of the cdf (20) of the Chaudhry and Zubair [1]'s GGIGD are given for some selected values of the parameters in Figure 2 (a – c) below:

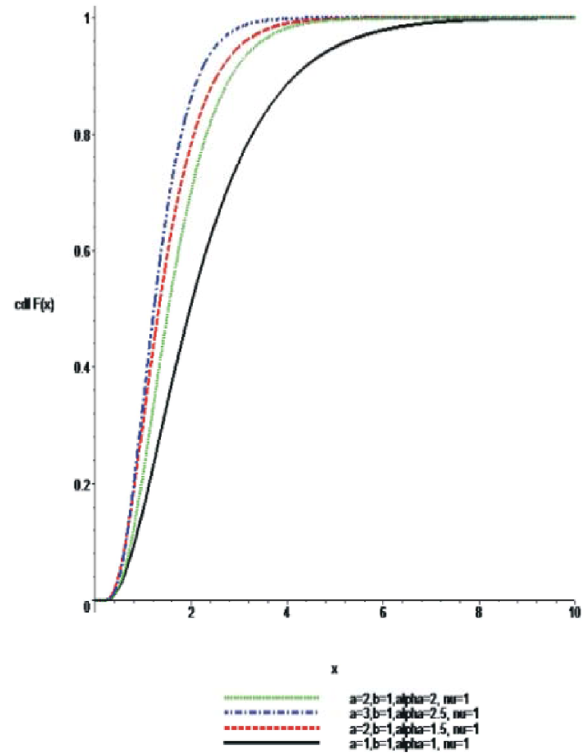
Reliability Analysis: As pointed out by Ahsanullah *et al*. [34], "the reliability analysis, such as the survival (or reliability) and the hazard (or failure rate) functions, of lifetime distributions play important roles in modelling many phenomena in the fields of biological, economics, engineering, physical and other pure and applied sciences. For a non-repairable population, we define the failure rate as the instantaneous rate of failure for the survivors to time *t* during the next instant of time". Therefore, in what follows, motivated by the importance of the reliability modelling of real data in the studies of the lifetime distributions, some reliability characteristics of the Chaudhry and Zubair's GGIGD are investigated. The reliability function $R(x)$ and the hazard functions $h(x)$ corresponding to the pdf (1) are respectively given by;

$$R(x) = 1 - F_X(x) = 1 - C\int_0^x t^{\alpha-1}\exp(-at)W_{0,\nu+\frac{1}{2}}\left(\frac{2b}{t}\right)dt, \tag{23}$$

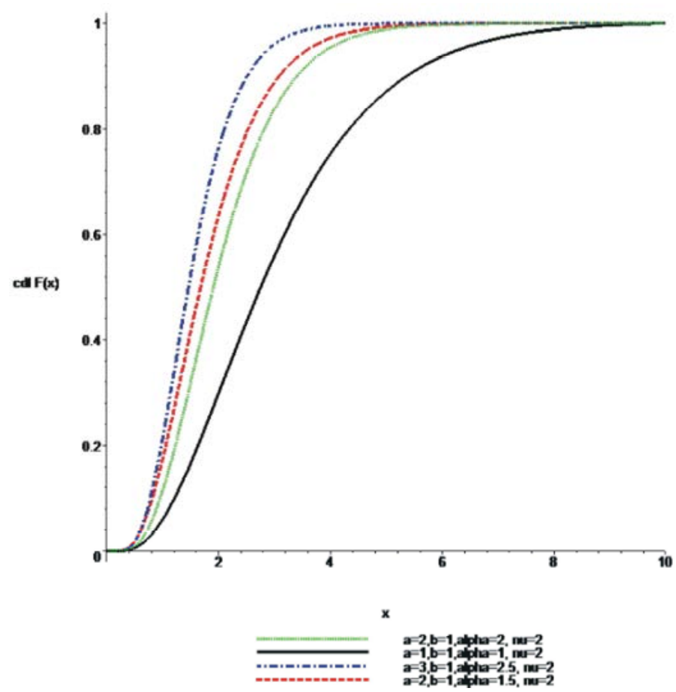
and

$$h(x) = \frac{f_X(x)}{1 - F_X(x)} == \frac{C x^{\alpha-1}\exp(-ax)W_{0,\nu+\frac{1}{2}}\left(\frac{2b}{x}\right)}{1 - C\int_0^x t^{\alpha-1}\exp(-at)W_{0,\nu+\frac{1}{2}}\left(\frac{2b}{t}\right)dt}. \tag{24}$$





(a) $\{\alpha = 1,1.5, 2,2.5; a = 1,2,3; b = 1; v = 0\}$ left); (b) $\{\alpha = 1,1.5, 2,2.5; a = 1,2,3; b = 1; v = 1\}$ (right); and



(c) $\{\alpha = 1,1.5, 2,2.5; a = 1,2,3; b = 1; v = 2\}$ (bottom)

Fig. 2: Plots of the Chaudhry and Zubair [1]'s GGIGD cdf (20) for:

It is obvious that the Eqs. (23) and (24) cannot be integrated analytically in closed forms and so should be evaluated numerically. Furthermore, since it is known from Gradshteyn and Ryzhik ([30], Eq. 9.235.2, P. 1062) that $W_{0,v}(2z) = \left(\frac{2z}{\pi}\right)^{1/2} K_v(z)$, hence, in view of this formula, the equations (22), (23) and (24) can be simplified in terms of the Macdonald function $K_v(z)$. Also, by using the definitions (20) and (21), the reliability function, $R(x)$ and the hazard functions, $h(x)$, corresponding to the pdf (1), are respectively given in terms of the generalizations of the generalized incomplete gamma functions as follows:

$$R(x) = C a^{-\alpha} \Gamma_v(\alpha, ax; ab), \tag{25}$$

and

$$h(x) = \frac{a^{\alpha} \exp(-ax) W_{0, v+\frac{1}{2}}\left(\frac{2b}{x}\right)}{\Gamma_v(\alpha, ax; ab)}. \tag{26}$$

Differentiating Eq. (24) with respect to x, we have,

$$\begin{aligned} h'(x) &= \frac{f'(x)}{f(x)} h(x) + [h(x)]^2 \\ &= \frac{\left[x^{\alpha-1} \exp(-ax) W_{0, v+\frac{1}{2}}\left(\frac{2b}{x}\right)\right]'}{x^{\alpha-1} \exp(-ax) W_{0, v+\frac{1}{2}}\left(\frac{2b}{x}\right)} \left(\frac{C x^{\alpha-1} \exp(-ax) W_{0, v+\frac{1}{2}}\left(\frac{2b}{x}\right)}{1 - C \int_0^x t^{\alpha-1} \exp(-at) W_{0, v+\frac{1}{2}}\left(\frac{2b}{t}\right) dt} \right) \\ &+ \left(\frac{C x^{\alpha-1} \exp(-ax) W_{0, v+\frac{1}{2}}\left(\frac{2b}{x}\right)}{1 - C \int_0^x t^{\alpha-1} \exp(-at) W_{0, v+\frac{1}{2}}\left(\frac{2b}{t}\right) dt} \right)^2, \end{aligned} \tag{27}$$

where $x > 0$, $v \geq 0$, a . 0, $b \geq 0$, $-\infty < \alpha < \infty$. To discuss the behavior of the failure rate function, $h(x)$, letting $h(x) = 0$ in Eq. (27), we observe that the nonlinear equation $h(x) = 0$ cannot be solved in a closed form. Therefore, it should be solved numerically by Newton-Raphson method of iteration using some mathematical software such as Maple, Mathematica or R. Furthermore, as we observe from the Eq. (27) that $h(x)$ is positive irrespective of the values of the parameters $\{\alpha, a, b, v\}$, it follows that the Chaudhry and Zubair's GGIGD has the increasing failure rate (IFR) property.

Possible Shapes of the Hazard Function of the Chaudhry and Zubair [1]'s GGIGD: For some special values of the parameters, the graphs of the hazard function (hf) (22) are illustrated in Figure 3 (a – c) below:

The effects of the parameters are obvious from these figures. The increasing and bathtub shape (both concave down and concave up) behaviors of the hazard function (hf), $h(x)$, are also evident from these Figures. Moreover, it is sometimes useful to find the average failure rate function (*AFR*), over any interval, say, $(0,t)$ that averages the failure rate over the interval, $(0,t)$, see, for example, Barlow and Proschan [35]. Thus, the average failure rate function (*AFR*) of the Chaudhry and Zubair's GGIG distribution, over the interval $(0, t)$, is given by;

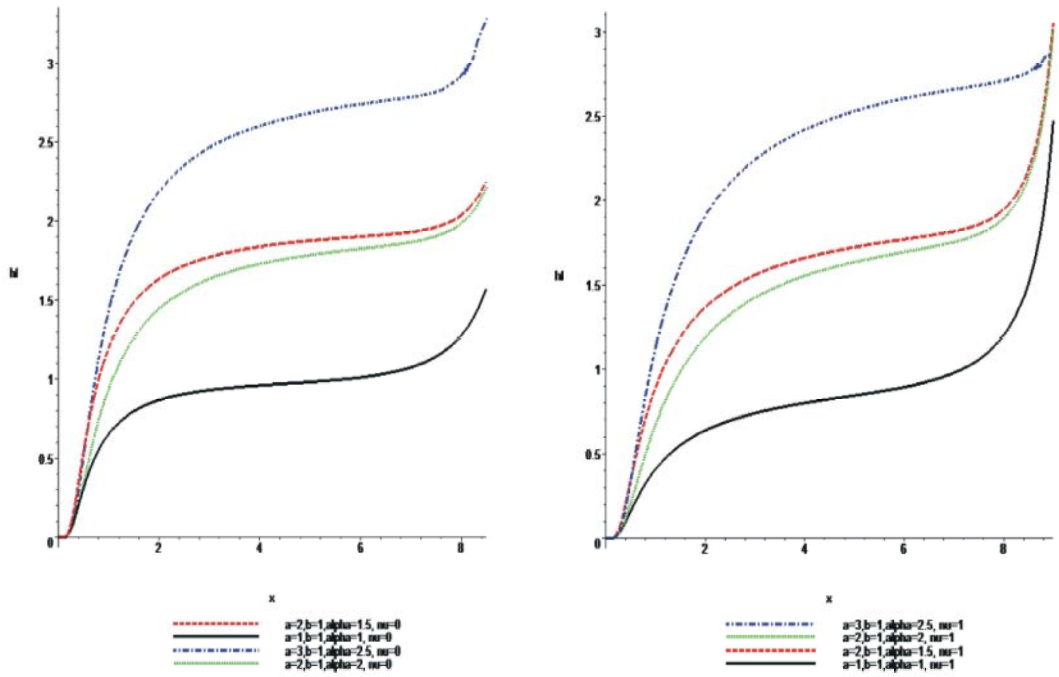
$$AFR = \frac{-\ln(R(t))}{t} = -\frac{1}{t} \ln\left(1 - C \int_0^t z^{\alpha-1} \exp(-az) W_{0, v+\frac{1}{2}}\left(\frac{2b}{z}\right) dz\right),$$

which in view of the expansion of logarithmic function as a power series, is seen to be positive irrespective of the values of the parameters $\{\alpha, a, b, v\}$. It follows that the Chaudhry and Zubair's GGIG distribution is increasing failure rate on average (IFRA). Furthermore, a life distribution $F(.)$ is new better than used (NBU) if $R(x + y) \leq R(x) R(y)$, $\forall x, y > 0$ and new worse than used (NWU) if the reversed inequality holds, see, for example, Barlow and Proschan [35]. We note that, for the Chaudhry and Zubair's GGIG distribution from the Eq. (21), since.

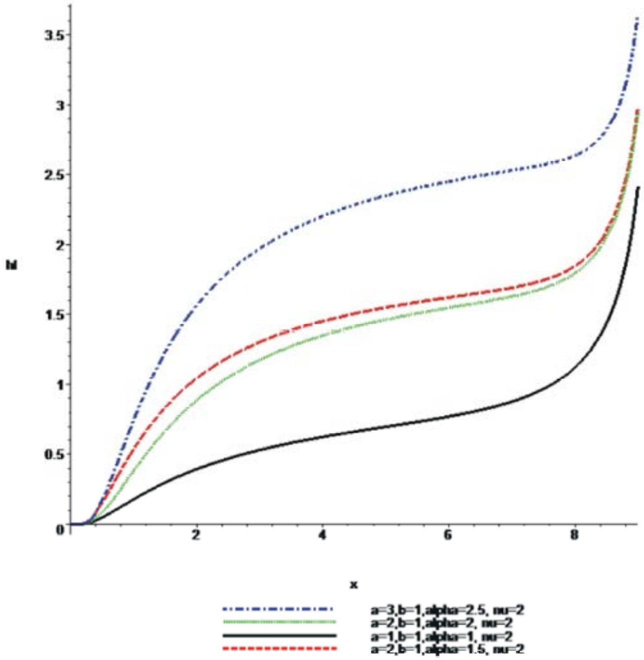
$$R(x+y) = C\,a^{-\alpha}\Gamma_v\left(\alpha, a(x+y); ab\right),$$

and

$$R(x).R(y) = \left[C\,a^{-\alpha}\Gamma_v\left(\alpha, ax; ab\right)\right]\times\left[C\,a^{-\alpha}\Gamma_v\left(\alpha, ay; ab\right)\right],$$



(a) $\{\alpha = 1,1.5, 2, 2.5; a = 1,2, 3; b = 1; v = 0\}$ (left); (b) $\{\alpha = 1,1.5, 2, 2.5; a = 1,2, 3; b = 1; v = 1\}$ (right);



and (c) $\{\alpha = 1,1.5, 2, 2.5; a = 1,2, 3; b = 1; v = 2\}$ (bottom).

Fig. 3: Plots of the Chaudhry and Zubair [1]'s GGIGD hf for:

It is easy see that $R(x + y) \leq R(x), R(y)$, which implies that the distribution of the Chaudhry and Zubair's GGIG distribution has the property of new better than used (NBU).

Remark 3.2: As pointed out by Chaudhry and Zubair ([1], p. 196), "the study of the cumulative distribution function (cdf) (20) and the reliability function (21) is important in statistics and reliability theory. In particular, the systematic study of these functions will extend the usefulness of the generalized inverse Gaussian distributions in reliability and life-testing situations with censored data; see Jorgensen [11]".

Moments: Here, we derive various moments of the Chaudhry and Zubair [1]'s GGIG distribution.

Moment: For positive integer k, the *kth* moment of the random variable X of the Chaudhry and Zubair [1]'s GGIG distribution is given by;

$$E\left(X^k\right) = (C)\int_0^{\infty} x^{\alpha+k-1} \exp(-ax) W_{0,\nu+\frac{1}{2}}\left(\frac{2b}{x}\right)dx, \tag{28}$$

where $x > 0$, $\nu \geq 0$, $a > 0$, $b \geq 0$, $-\infty < \alpha < \infty$ and

$$C = \left(\frac{1}{a^{\alpha-\frac{1}{2}}} 2^{\alpha-2}\pi^{-1}(ab)^{1/2} G_{0,4}^{4,0}\left(\frac{(ab)^2}{16}\middle| \frac{1}{2}\left(\nu+\frac{1}{2}\right), \frac{-1}{2}\left(\nu+\frac{1}{2}\right), \frac{1}{2}\left(\alpha+\frac{1}{2}\right), \frac{1}{2}\left(\alpha-\frac{1}{2}\right)\right)\right)^{-1}. \tag{29}$$

Using the Eq. 4.14, P. 197 of Chaudhry and Zubair [1], the *kth* moment is easily given by the following formula:

$$E\left(X^k\right) = (C)\frac{1}{a^{\alpha+k-\frac{1}{2}}} 2^{\alpha+k-2}\pi^{-1}(ab)^{1/2} G_{0,4}^{4,0}\left(\frac{(ab)^2}{16}\middle| \frac{1}{2}\left(\nu+\frac{1}{2}\right), \frac{-1}{2}\left(\nu+\frac{1}{2}\right), \frac{1}{2}\left(\alpha+k+\frac{1}{2}\right), \frac{1}{2}\left(\alpha+k-\frac{1}{2}\right)\right), \tag{30}$$

where C is given as mentioned above and $G_{0,4}^{4,0}(.)$ denotes the Meijer G-function; see, for example, Mathai [29].

First Moment (or Mean): Taking $k = 1$, in the above-mentioned equation of the *kth* moment, the mean (or the first moment) of the random variable X is easily given by;

$$E(X) = (C)\frac{1}{a^{\alpha+\frac{1}{2}}} 2^{\alpha-1}\pi^{-1}(ab)^{1/2} G_{0,4}^{4,0}\left(\frac{(ab)^2}{16}\middle| \frac{1}{2}\left(\nu+\frac{1}{2}\right), \frac{-1}{2}\left(\nu+\frac{1}{2}\right), \frac{1}{2}\left(\alpha+\frac{3}{2}\right), \frac{1}{2}\left(\alpha+\frac{1}{2}\right)\right) \tag{31}$$

***kth* (Central) Moment:** The *kth* (central) moment, β_k, of the random variable X of the Chaudhry and Zubair [1]'s GGIG distribution can easily be derived as follows:

$$\begin{aligned} \beta_k &= E\left[X - E(X)\right]^k = \int_0^{\infty} \left[x - E(X)\right]^k f_X(x)\,dx \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(E(X)\right)^j E\left(X^{k-j}\right), \end{aligned} \tag{32}$$

where $(E(X))_j$ and $E(X^{k-j})$ can be obtained from the equations (31) and (30), respectively. From the equation (32), one can easily obtain the second, third and higher central moments.

Variance: Taking $k = 2$ in equation (32), the variance is given by,

$$\beta_2 = E\left[X - E(X) \right]^2 = \int_0^\infty \left[x - E(X) \right]^2 f_X(x)\,dx = E\left[X^2\right] - \left(E[X]\right)^2$$

$$= (C)\left[\frac{1}{a^{\alpha+\frac{3}{2}}} 2^{\alpha} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0}\left(\frac{(ab)^2}{16} \middle| \frac{1}{2}\left(\nu+\frac{1}{2}\right), \frac{-1}{2}\left(\nu+\frac{1}{2}\right), \frac{1}{2}\left(\alpha+\frac{5}{2}\right), \frac{1}{2}\left(\alpha+\frac{3}{2}\right) \right) \right. \tag{33}$$

$$\left. -(C)\left(\frac{1}{a^{\alpha+\frac{1}{2}}} 2^{\alpha-1} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0}\left(\frac{(ab)^2}{16} \middle| \frac{1}{2}\left(\nu+\frac{1}{2}\right), \frac{-1}{2}\left(\nu+\frac{1}{2}\right), \frac{1}{2}\left(\alpha+\frac{3}{2}\right), \frac{1}{2}\left(\alpha+\frac{1}{2}\right) \right) \right)^2 \right].$$

Coefficients of Skewness and Kurtosis: By taking $k = 3$ and $k = 4$ in the equation (32), the third and fourth central moments, β_3 and β_4, are respectively obtained as follows:

$$\beta_3 = \int_0^\infty \left[x - E(X) \right]^3 f_X(x)\,dx = \sum_{j=0}^{3} (-1)^j \binom{3}{j} \left(E(X) \right)^j E\left(X^{3-j} \right), \tag{34}$$

and

$$\beta_4 = \int_0^\infty \left[x - E(X) \right]^4 f_X(x)\,dx = \sum_{j=0}^{4} (-1)^j \binom{4}{j} \left(E(X) \right)^j E\left(X^{4-j} \right). \tag{35}$$

Thus, using the above-mentioned expressions for β_3 and β_4, the measure of skewness, γ_1 and kurtosis, γ_2, are respectively given by;

$$\gamma_1 = \frac{\beta_3}{(\beta_2)^{3/2}} = \frac{\sum_{j=0}^{3} (-1)^j \binom{3}{j} \left(E(X) \right)^j E\left(X^{3-j} \right)}{\left(E\left[X^2\right] - E[X]^2 \right)^{\frac{3}{2}}}, \tag{36}$$

and

$$\gamma_2 = \frac{\beta_4}{(\beta_2)^2} = \frac{\sum_{j=0}^{4} (-1)^j \binom{4}{j} \left(E(X) \right)^j E\left(X^{4-j} \right)}{\left(E\left[X^2\right] - E[X]^2 \right)^2}, \tag{37}$$

where $(E(X))_j$ and $E(X^{k-j})$ can be obtained from the equations (31) and (30), respectively.

Using the software Maple in the above-mentioned equations, the numerical values of the first moment $E(X)$, variance β_2, skewness γ_1 and kurtosis γ_2 are tabulated in the Tables 1-3 for some selected values of the parameters.

Table 1: When $\nu = 0$

	Moments					
Parameters	$E(X)$	β_2	β_3	β_4	γ_1	γ_2
a=1, b=1, $\alpha = 1$	1.8146	1.3371	2.4506	12.3991	1.5850	6.9352
a=2, b=1, $\alpha = 2$	1.4517	0.5702	0.5386	1.7633	1.2510	5.4240
a=2, b=1, $\alpha = 1.5$	1.2724	0.4715	0.4350	1.2940	1.3429	5.8208
a=3, b=1, $\alpha = 2.5$	1.1850	0.3117	0.1970	0.4840	1.1316	4.9815

Table 2: When $\nu = 1$

	Moments					
Parameters	$E(X)$	β_2	β_3	β_4	γ_1	γ_2
a=1, b=1, $\alpha = 1$	2.2916	1.9842	3.9155	23.5740	1.4009	5.9785
a=2, b=1, $\alpha = 1$	1.6842	0.7360	0.7320	2.7274	1.1600	5.040
a=2, b=1, $\alpha = 1.5$	1.4800	0.6200	0.6095	2.0674	1.2492	5.3825
a=3, b=1, $\alpha = 2.5$	1.3276	0.3814	0.2522	0.6900	1.0707	4.7418

Table 3: When $v = 2$

	Moments					
Parameters	$E(X)$	β_2	β_3	β_4	γ_1	γ_2
a=1, b=1, $\alpha = 1$	3.1030	2.9652	5.8590	43.9553	1.1475	4.9993
a=2, b=1, $\alpha = 2$	2.0711	0.9800	0.9900	4.3745	1.0210	4.5560
a=2, b=1, $\alpha = 1.5$	1.8422	0.8537	0.8620	3.4892	1.0928	4.7874
a=3, b=1, $\alpha = 2.5$	1.5706	0.4877	0.3287	1.0450	0.9651	4.3927

It is observed from the above-mentioned computations that the skewness, γ_1, is positive. Hence, the Chaudhry and Zubair's GGIG distribution of the random variable X is positively skewed. Furthermore, based on our calculations, we observe that the kurtosis, $\gamma_2 > 3$. Thus, the Chaudhry and Zubair's GGIGD is heavy tailed.

Moment Generating Function, Characteristic Function and *rth* Cumulant: It is easy to see that, for the Chaudhry and Zubair's GGIG distribution, the moment generating and characteristic functions of the random variable X are respectively given by;

$$M_X(t) = E\left(e^{iX}\right) = \sum_{k=0}^{\infty} \frac{(t)^k}{k!} E\left(X^k\right), \tag{38}$$

and

$$M_X(it) = E\left(e^{itX}\right) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E\left(X^k\right), \tag{39}$$

where $i = \sqrt{-1}$ is the imaginary number, $i^2 = -1$ and $E(X^k)$ denotes the *kth* moment about the origin of the random variable X which can easily obtained from the equation (30). The *rth* cumulant, κ_r, of the random variable X having the characteristic function (39) is given by;

$$\kappa_r = \frac{1}{i^r}\left[\frac{d^r\left(\ln f_X(t)\right)}{dt^r}\right]_{t=0}, \quad r = 1, 2, \ldots, \tag{40}$$

from which, by successive differentiation, it can be easily seen that.

$\kappa_1 = E(X) = \alpha_1$, $\kappa_2 = Var(X) = \beta_2$, $\kappa_3 = E\,[X - E\,(X)]^3 = \beta_3$ etc.,

which can easily obtained by using the equations (31), (32) and (33).

First Incomplete Moment: The first incomplete moment of the random variable X is given as;

$$I_X(x) = \int_0^x u\, f(u)\, du = C\int_0^x u \exp(-au) W_{0, v+\frac{1}{2}}\left(\frac{2b}{u}\right) du$$
$$= C\gamma_v(2, ax; ab),$$

Using the definition (18) of the generalizations of the generalized incomplete gamma function; see Chaudhry and Zubair ([1], Eq. 4.8, p. 196).

Shannon Entropy: The Shannon entropy measure of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics, among others. According to Shannon [36], the entropy measure of a continuous real random variable X is defined as;

$$H_X\left[f_X(X)\right] = E[-\ln(f_X(X)] = -\int_{-\infty}^{\infty} f_X(x) \ln\left[f_X(x)\right] dx.$$

Thus, following the above-mentioned formula, the entropy of the Chaudhry and Zubair's GGIG distribution is obtained as follows:

$$H_X\left[f_X(X)\right] = -\int_0^{\infty}\left[\ln\left(C x^{\alpha-1} e^{-ax} W_{0,v+\frac{1}{2}}\left(\frac{2b}{x}\right)\right)\right]\left[C x^{\alpha-1} e^{-ax} W_{0,v+\frac{1}{2}}\left(\frac{2b}{x}\right)\right] dx$$

$$= a E(X) - (\alpha-1) E\left(\ln(X)\right) - E\left(\ln\left(W_{0,v+\frac{1}{2}}\left(\frac{2b}{X}\right)\right)\right) - \ln(C), \tag{41}$$

where $x > 0$, $v \geq 0$, $a > 0$, $b \geq 0$, $-\infty < \alpha < \infty$ and

$$C = C(\alpha; a, b, v) = \left(\int_0^{\infty} x^{\alpha-1} \exp(-ax) W_{0,v+\frac{1}{2}}\left(\frac{2b}{x}\right) dx\right)^{-1} \tag{42}$$

Denotes the normalizing constant and $W_{k,v}(z)$ denotes the Whittaker function for reals κ and v and real argument z. It is obvious that the Eq. (41) cannot be evaluated in a closed form. Hence, one should use some computer packages such as Maple, or Mathematica, or R, or MathCAD14, or other software, to tabulate the values of the Shannon entropy measure for different values of the parameters.

Remark 3.1 (Special Cases): Here, we will derive some special cases of the Shannon entropy of Chaudhry and Zubair's GGIG distribution as follows:

Case (I) (when $v = 0$): As mentioned in Section 2, when $v = 0$, the pdf in the Eq. (9) is given by;

$$f_X(x) = C x^{\alpha-1} e^{-\frac{b}{x} - ax}, x > 0, a > 0, b > 0, -\infty < \alpha < \infty,$$

where $C = \frac{1}{2}\left(\frac{a}{b}\right)^{\frac{\alpha}{2}} \frac{1}{K_\alpha\left(2\sqrt{ab}\right)}$ is the normalizing constant. Using the Shannon entropy formula to the above-mentioned expression for the pdf and applying the Gradshteyn and Ryzhik ([30], Eq. 3.471.9, p. 340) and simplifying, it is easily seen that the expression for the entropy is obtained as follows:

$$H_X\left[f_X(X)\right] = -\int_0^{\infty}\left[\ln C x^{\alpha-1} e^{-\frac{b}{x} - ax}\right]\left[C x^{\alpha-1} e^{-\frac{b}{x} - ax}\right] dx$$

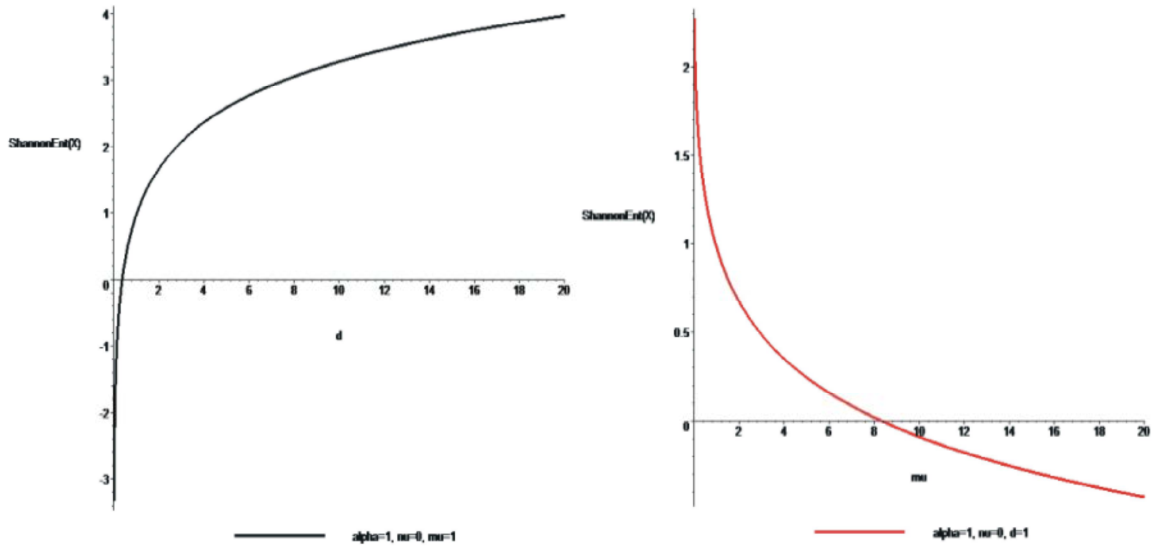
$$= \left(\sqrt{ab}\right)\left[\frac{K_{\alpha-1}\left(2\sqrt{ab}\right) + K_{\alpha+1}\left(2\sqrt{ab}\right)}{K_\alpha\left(2\sqrt{ab}\right)}\right] - (\alpha-1) E\left(\ln(X)\right) - \ln(C). \tag{43}$$

Case (II) (when $v = 0$, $\alpha = 1$): Taking $\alpha = 1$ in the Eq. (43), the Shannon entropy is given by;

$$H_X\left[f_X(X)\right] = \left(\sqrt{ab}\right)\left[\frac{K_0\left(2\sqrt{ab}\right) + K_2\left(2\sqrt{ab}\right)}{K_1\left(2\sqrt{ab}\right)}\right] - \ln(C). \tag{44}$$

Case (III) (when $v = 0, \alpha = 0, a = \frac{\mu}{d}, b = \mu d, \mu > 0, d > 0$): In this case, the pdf (1) of the Chaudhry and Zubair's GGIG distribution reduces to the following:

$$f_X(x) = \frac{1}{2 K_0(2\mu)} x^{-1} e^{-\mu\left(\frac{d}{x} + \frac{x}{d}\right)}, x > 0, \mu > 0, d > 0. \tag{45}$$



(a) $\{\alpha = 0; \nu = 0; \mu = 1\}$ (left); and (b) $\{\alpha = 0; \nu = 0; d = 1\}$(right)

Fig. 4: Shannon Entropy Plots of the Chaudhry and Zubair [1]'s GGIGD for:

Using the Shannon entropy formula to the above-said pdf (45), applying the Gradshteyn and Ryzhik ([30], Eq. 4.356.1, p. 577), that is, $\int_0^\infty e^{-\mu\left(\frac{d}{x}+\frac{x}{d}\right)} x^{-1}\ln(x)\,dx = 2\ln(d)K_0(2\mu)$ and the Gradshteyn and Ryzhik ([30], Eq. 3.471.9, p. 340) and noting that $K_{-\nu} \equiv K_\nu$ (Eq. 3.471.12, p. 340, Gradshteyn and Ryzhik 3[0], it is easily seen, after simplifying, that the expression for the Shannon entropy in this case is given by;

$$H_X\left[f_X(X)\right] = 2\mu\frac{K_1(2\mu)}{K_0(2\mu)} + \ln\left[2d\,K_0(2\mu)\right], d > 0, \mu > 0. \tag{46}$$

For some values of the parameters, using the Maple software, the graphs of the Shannon entropy (41) are sketched in Figure 4 (a – b) below. From Figure 4 (a), it is observed that it is a monotonic increasing and concave down function of d. From Figure 3.4 (b), we observe that it is a monotonic decreasing and concave up function of μ.

Percentile Points: Before any statistical applications of a given distribution, it is also important to know the percentile points. For example, we may be interested in knowing the median (50%), 25%, or 75% quartiles. Similarly, it is necessary to compute the 90%, 95%, or 99% confidence levels for other applications in order to assess the statistical significance of an observation whose distribution is known. Thus, in view of these facts, in what follows, we have computed the percentage points of the Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution (GGIGD).

The 100 *pth* percentile or the quantile of order p, for any $0 < p < 1$, of the Chaudhry and Zubair [1]'s GGIGD, with the pdf f_x (x) as in Eq. (1) is defined as a number x_p such that the area under $f_X(x)$ to the left of x_p is p. In other words, x_p is any solution of the equation $F(x_p) = \int_1^{x_p} f_X(u)\,du = p,$, where f_X (x) denotes the cdf given by Eq. (20). Thus, using a Maple program, we have numerically solved the equation $F(x_p) = \int_1^{x_p} f_X(u)\,du = p,$ and computed the percentage points x_p associated with the cdf, $F(x_p)$, of X for different sets of values of the parameters, which are provided in the Table 4 below.

Table 4: Percentile Points of the Chaudhry and Zubair [1]'s GGIGD, $X \sim$ GGIGD (α, a, b, v)

		Percentiles p					
Parameters		0.75	0.80	0.85	0.90	0.95	0.99
$v = 0$, $\alpha = 1$ $a = 1$, $b = 1$	x_p	2.34291	2.58963	2.90240	3.33592	4.06376	5.71303
$v = 0$, $\alpha = 1.5$ $a = 2$, $b = 1$	x_p	1.61441	1.75828	1.93847	2.18494	2.59234	3.49920
$v = 0$, $\alpha = 2$ $a = 2$, $b = 1$	x_p	1.83945	1.99661	2.19245	2.45876	2.89566	3.85731
$v = 0$, $\alpha = 2.5$ $a = 3$, $b = 1$	x_p	1.48221	1.59689	1.73893	1.93079	2.24295	2.92190
$v = 1$, $\alpha = 1$ $a = 1$, $b = 1$	x_p	2.97570	3.27426	3.64846	4.16002	5.00244	6.83466
v, $= 1$, $\alpha = 1.5$ $a = 2$, $b = 1$	x_p	1.88497	2.04936	2.25406	2.53212	2.98745	3.98598
$v = 1$, $\alpha = 2$ $a = 3$, $b = 1$	x_p	2.13834	2.31583	2.53580	2.83296	3.31630	4.36588
$v = 1$, $\alpha = 2.5$ $a = 3$, $b = 1$	x_p	1.6629	1.78922	1.94509	2.15469	2.49367	3.22385
$v = 2$, $\alpha = 1$ $a = 1$, $b = 1$	x_p	3.98335	4.33629	4.77200	5.35660	6.29117	8.14912
$v = 2$, $\alpha = 1$ $a = 2$, $b = 1$	x_p	2.34191	2.53183	2.76627	3.08158	3.59156	4.68975
$v = 2$, $\alpha = 2$ $a = 2$, $b = 1$	x_p	2.61738	2.81890	3.06672	3.39874	3.93310	5.07558
$v = 2$, $\alpha = 2.5$ $a = 3$, $b = 1$	x_p	1.96161	2.10264	2.27564	2.50674	2.87731	3.66547

Estimation of Parameters of the Chaudhry and Zubair [1]'s GGIGD: In what follows, we provide the estimation of the parameters $\{\alpha, a, b, v\}$ of the Chaudhry and Zubair (2002)'s GGIGD.

The Method of Moments: If $\{X_i\}_{i=1}^{n}$ be an *iid* sample from a distribution with a *m*-dimensional parameter vector ϕ, then, according to the method of moment (MOM), the estimator ϕ is the solution of the following system of equations:

$$E_{\tilde{\phi}}\left(X^k\right) = \frac{\sum_{i=1}^{n} X_i^{\,k}}{n}, \quad k = 1, 2, 3, \ldots, m \tag{47}$$

Thus, using the above-mentioned definition of MOM (5.1), we can obtain the first four moments from the Eq. (30) of the *kth* moment, $E(X)^k$, of the Chaudhry and Zubair's GGIG distribution by taking k = 1,2,3,4 and evaluating the respective expressions of the first four moments numerically. Then, the moment estimations of the parameters $\{\alpha, a, b, v\}$ can be determined by solving the system of four equations thus obtained by Newton-Raphson's iteration method and using some computer packages such as Maple, or Mathematica, or R, or MathCAD, or other software.

The Method of Maximum Likelihood: Given a sample $\{x_i\}$, i = 1,2,3,...,n, the likelihood function of the Chaudhry and Zubair's GGIGD pdf (1) is given by $L = \prod_{i=1}^{n} f\left(x_i\right)$. The objective of the likelihood function approach is to determine those values of the parameters that maximize the function L. Suppose $R = \ln\left(L\right) = \sum_{i=1}^{n} \ln\left[f\left(x_i\right)\right]$. Then, upon differentiation, the maximum likelihood estimates (MLE) of the parameters $\{\alpha, a, b, v\}$ can be obtained by solving the maximum likelihood system of equations.

$$\frac{\partial R}{\partial \alpha}=0,\ \frac{\partial R}{\partial a}=0,\ \frac{\partial R}{\partial b}=0 \text{ and } \frac{\partial R}{\partial v}=0, \tag{48}$$

Applying the Newton-Raphson's iteration method and using some computer packages such as Maple, or Mathematica, or R, or MathCAD14, or other software.

Remark 5.1: According to Balakrishnan and Chen ([27], p.2), "the estimation problem associated with the three-parameter inverse Gaussian distribution is a difficult and challenging one". As pointed out by Shakil *et al*. [37], when $\alpha = -\frac{1}{2}$, the maximum likelihood estimates of the parameters a and b can be found in Koutrouvelis *et al*. [38]. For the maximum likelihood estimates of the parameters a, b and α of the generalized inverse Gaussian (GIG) distribution, the interested readers are referred to Jorgensen [11]. For a three-parameter inverse Gaussian distribution, see Cheng and Amin [32]. Thus, in view of these facts, the maximum likelihood estimates of the parameters $\{\alpha, a, b, v\}$ of the Chaudhry and Zubair's GGIGD model may be possible to determine by solving the system of equations (48) by developing some iteration methods and using some computer software, such as R, Maple, Mathematica, MathCAD 14, etc. Thus, since the Chaudhry and Zubair's GGIGD pdf (1) involves the Whittaker function and also as remarked by Chaudhry and Zubair ([1]. p. 196), " the systematic study of these functions will extend the usefulness of the generalized inverse Gaussian distributions in reliability and life-testing situations with censored data", it is conjectured that the estimation of the parameters $\{\alpha, a, b, v\}$ of the Chaudhry and Zubair's GGIGD distribution and its applications to real life-time data are also a daunting task and are one of the major areas for further research.

Applications (Goodness of Fit Test of the Chaudhry and Zubair [1]'s GGIGD): In this section, the goodness of fit test of the Chaudhry and Zubair's GGIGD vis-à-vis gamma and lognormal distributions will be provided by considering two real-world data examples.

Example 1: This example considers a random sample of the female white blood cell count (1000 cells /μL measured for 40 different aged adult females as reported in Triola ([39], p. 593), which are provided in Table 5. Based on this example, we test the chi-squared goodness-of-fit of the Chaudhry and Zubair's GGIG distribution to this data and compare it with the gamma and lognormal distributions.

Table 5: Female White Blood Cell Count (1000 cells /μL Measured for 40 Different Aged Adult Females

9.6, 7.1, 7.5, 6.8, 5.6, 5.4, 6.7, 8.6, 10.2, 4.1, 13.0, 9.2, 5.9, 8.0, 7.0, 9.1, 5.7, 4.6, 6.0, 5.7, 8.9, 6.4, 8.1, 7.9, 4.4, 4.9, 5.3, 5.3, 4.7, 9.8, 5.3, 4.9, 6.3, 5.4, 7.0, 13.5, 10.0, 10.3, 5.1, 6.6

The descriptive statistics of the above-mentioned white blood cell count data are computed in Table 6. Furthermore, using the software minitab and statdisk, we have drawn the histogram and normal quantile plot of the data, which are given in Figure 5. Moreover, the Ryan-Joiner Test (Similar to Shapiro-Wilk Test) of normality assessment of the white blood cell count data is provided in Table 7.

Table 6: Descriptive Statistics: Female White Blood Cell Count

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
Blood Cell Count	40	0	7.148	0.360	2.276	4.100	5.325	6.650	8.75	13.500
Variable	Mode		Skewness		Kurtosis		Excess Kurtosis			
Blood Cell Count	5.3		1.0171		3.9602		0.7148			

Table 7: Ryan-Joiner Test (Similar to Shapiro-Wilk Test) of Normality Assessment

Ryan-Joiner Test
Test statistic, Rp: 0.9566
Critical value for 0.05 significance level: 0.9715
Critical value for 0.01 significance level: 0.9597
Reject normality with a 0.05 significance level.
Reject normality with a 0.01 significance level.
Possible Outliers
Number of data values below Q1 by more than 1.5 IQR: 0
Number of data values above Q3 by more than 1.5 IQR: 0

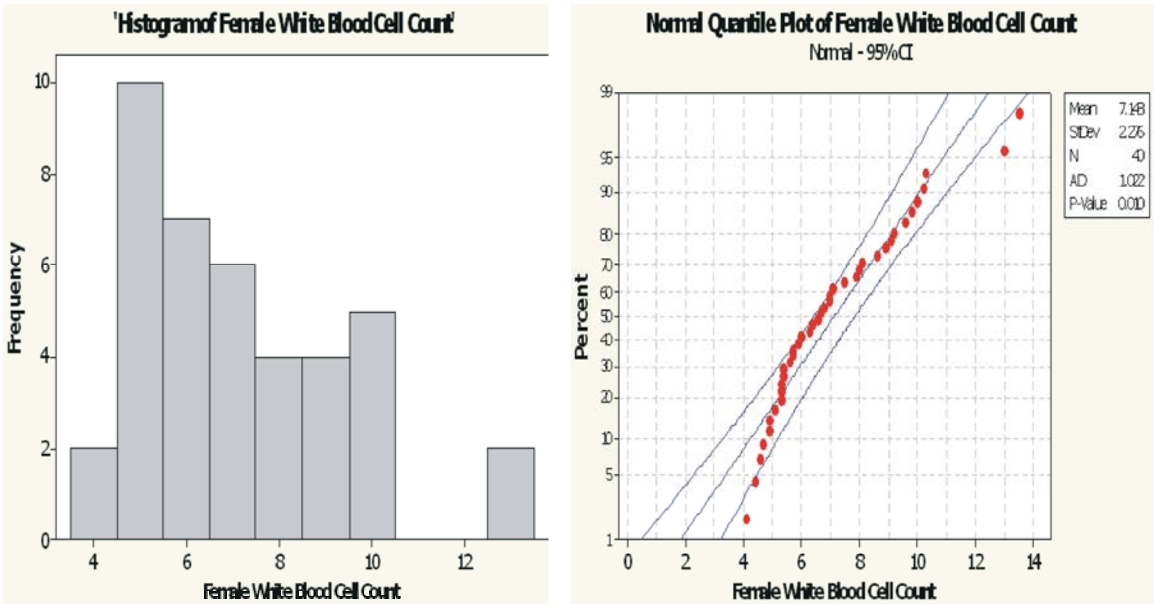


Fig. 5: Histogram (left); Normality Assessment (right)

From the Ryan-Joiner Test of Normality Assessment (Table 7) and Figure 5 (histogram and normal quantile plot), it is obvious that the shape of the blood cell count data is skewed to the right and heavy tailed. This is also confirmed from the skewness (1.0171) and kurtosis (3.9602) of the blood cell count data as computed in Table 6. Since fitting of a probability distribution to adult female white blood cell count data may be helpful in predicting the probability or forecasting the frequency of occurrence of the adult female white blood cell count, this suggests that 'y', the adult female white blood cell count data, could possibly be modeled by some skewed distributions. As such we have tested the fitting of the Chaudhry and Zubair's GGIG distribution, gamma and lognormal distributions based on their goodness of fit to the adult female white blood cell count data (as given in Table 5). For this, Maple 11 has been used for computing the data moments, estimating the parameters and chi-square test for goodness-of-fit. The data moments computed are given as follows:

$$\hat{\mu}_1 = 7.1475, \hat{\mu}_2 = 56.1353, \hat{\mu}_3 = 484.4956, \text{ and } \hat{\mu}_4 = 4563.4431.$$

The estimation of the parameters and chi-square goodness-of-fit test are provided in Tables 8 and 9 respectively.

Table 8: Parameter Estimates for the Female White Blood Cell Count Measurements Data Assuming Different Models

Chaudhry and Zubair [1]'s GGIGD *X ~ GGIGD* (*a*, *a*, *b*, *v*)	*X ~ Lognormal* (μ, σ^2) (where $-\infty < \mu < +\infty$ and $\sigma > 0$)	*X ~ Gamma (*k,, θ), (where shape $k > 0$ and rate $\theta > 0$)
GGIGD Model 1 ($\hat{v} = 0, \hat{\alpha} = -0.5, \hat{a} = 0.7080, \hat{b} = 36.1635$)	$\hat{\mu} = 1.9196, \hat{\sigma} = 0.3070$	$\hat{k} = 10.1192, \hat{\theta} = 1.4160$
GGIGD Model 2 ($\hat{v} = 0, \hat{\alpha} = 1, \hat{a} = 1, \hat{b} = 40.7537$)		
GGIGD Model 3 ($\hat{v} = 0, \hat{\alpha} = 2, \hat{a} = 1, \hat{b} = 33.9140$)		

Table 9: Comparison criteria (chi-square test for goodness-of-fit at the level of significance = 0.05)

	Model				
	GGIGD Model 2	GGIGD Model 3	Lognormal	GGIGD Model 1	Gamma
Test statistic	0.3186827	0.3576475	0.4925972	0.5021621	1.1325701
Critical value	5.9914645	5.9914645	5.9914645	5.9914645	5.9914645
P-value	0.8527052	0.8362533	0.7816888	0.7779599	0.5676304
GOF Fitting Rank	First	Second	Third	Fourth	Fifth

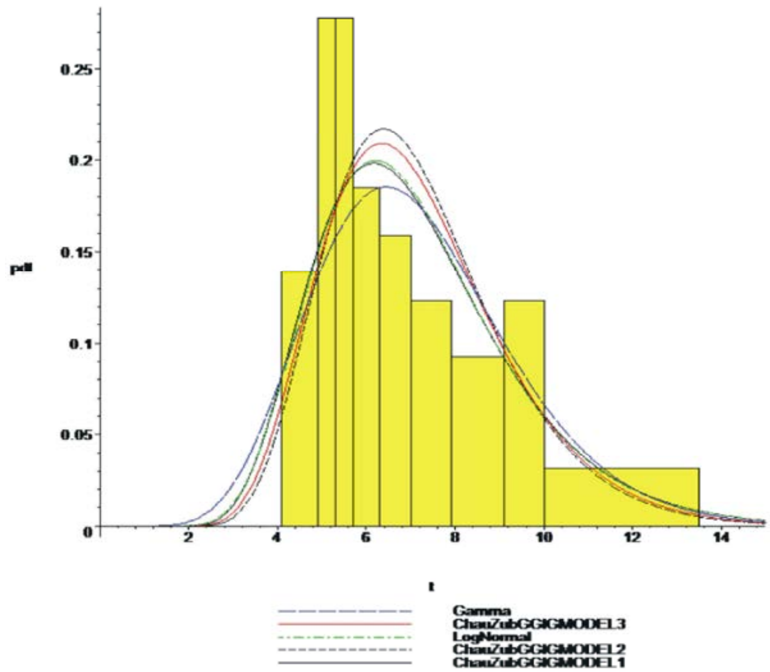


Fig. 5 (a) : Fitting of the pdfs of the Chaudhry and Zubair (2002) (GGIGD 1, 2 and 3), Lognormal and Gamma distributions to the Female White Blood Cell Count Measurements Data

From the chi-square goodness-of-fit test, we observed that the Chaudhry and Zubair [1]'s models (GGIGD 1, 2 and 3), lognormal and gamma distributions fit the female white blood cell count measurements data reasonably well. However, the Chaudhry and Zubair (2002)'s GGIGD Model 2 produces the highest p-value and smallest test statistic and therefore fitted better than GGIGD Model 1, GGIGD Model 3, lognormal and gamma distributions. Moreover, for the parameters estimated in Table 8, the Chaudhry and Zubair [1]'s models (GGIGD 1, 2 and 3), lognormal and gamma distributions have been superimposed on the histogram the female white blood cell count measurements data as shown in Figure 5 (a), from which we observed that the Chaudhry and Zubair [1]'s GGIGD Model 2 fits the female white blood cell count measurements data reasonably well.

Example 2: This example considers a random sample of 20 tree circumferences (in feet) as reported in Triola & Triola (P. 85, 2006), which are provided in Table 10. Based on this example, we tested the chi-squared goodness-of-fit of the three different cases of the Chaudhry and Zubair's GGIG distribution and compared it with the gamma and lognormal distributions to this data.

Table 10: A random sample of 20 tree circumferences (in feet) as reported in Triola & Triola (P. 85, 2006)

1.8, 1.9, 1.8, 2.4, 5.1, 3.1, 5.5, 5.1, 8.3, 13.7, 5.3, 4.9, 3.7, 3.8, 4.0, 3.4, 5.2, 4.1, 3.7, 3.9

The descriptive statistics of the above-mentioned tree circumference data are computed in Table 11. Furthermore, using the software statdisk and minitab, we have tested the normality of the tree circumference data by Ryan-Joiner Test (Similar to Shapiro-Wilk Test), along with drawing a histogram of the data, which are given in Figure 7 and Table 12 below.

Table 11: Descriptive Statistics: Tree Circumferences

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
Tree_Circumferen	20	0	4.535	0.593	2.651	1.800	3.175	3.950	5.15	13.700
Variable	Skewness				Kurtosis		Excess Kurtosis			
Tree_Circumferen	2.35				10.89		7.35			

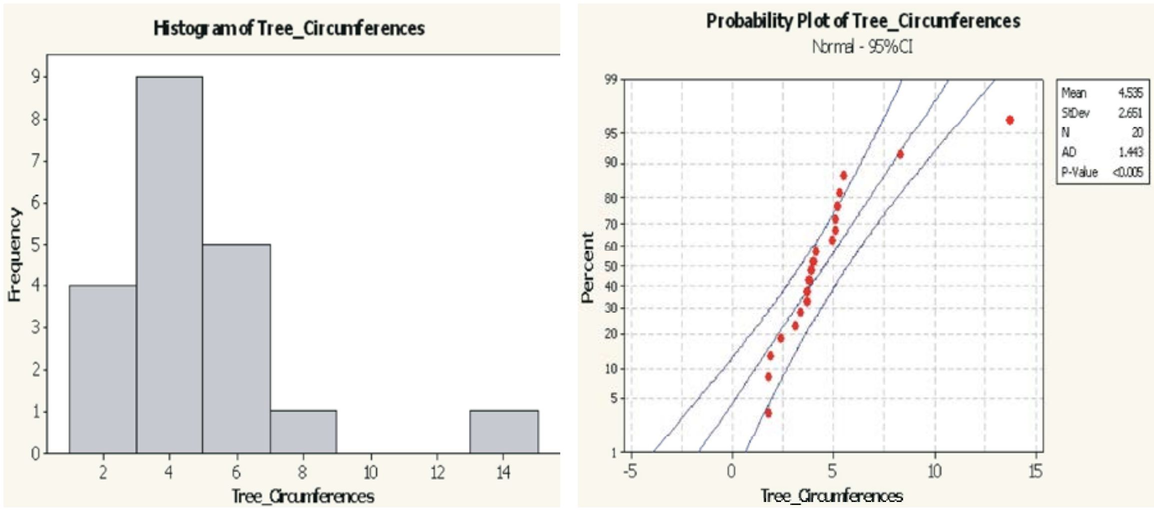


Fig. 6: (left): Histogram; (right): Normality Assessment

Table 12: Ryan-Joiner Test of Normality Assessment

Ryan-Joiner Test
Test statistic, Rp: 0.8596
Critical value for 0.05 significance level: 0.951
Critical value for 0.01 significance level: 0.928
Reject normality with a 0.05 significance level.
Reject normality with a 0.01 significance level.
Possible Outliers
Number of data values below Q1 by more than 1.5 IQR: 0
Number of data values above Q3 by more than 1.5 IQR: 2

Table 13: Parameter Estimates for the Tree Circumference Data Assuming Different Models

Chaudhry and Zubair [1]'s GGIGD $X \sim GGIGD\ (a, a, b, v)$	$X \sim Lognormal\ (\mu, \sigma^2)$ (where $-\infty < \mu < +\infty$ and $\sigma > 0$)	$X \sim Gamma\ (k, \theta)$, (where shape $k > 0$ and rate $\theta > 0$)
GGIGD Model 1 ($\hat{v} = 0, \hat{\alpha} = -0.5, \hat{a} = 0.3396, \hat{b} = 6.9850$)	$\hat{\mu} = 1.3713, \hat{\sigma} = 0.5302$	$\hat{k} = 3.0805, \hat{\theta} = 0.6793$
GGIGD Model 2 ($\hat{v} = 0, \hat{\alpha} = 1, \hat{a} = 1, \hat{b} = 14.1596$)		
GGIGD Model 3 ($\hat{v} = 0, \hat{\alpha} = 2, \hat{a} = 1, \hat{b} = 9.9715$)		

Table 14: Comparison criteria (chi-square test for goodness-of-fit at the level of significance = 0.05)

	Model				
	GGIGD Model 2	GGIGD Model 3	Lognormal	GGIGD Model 1	Gamma
Test statistic	2.7665	2.8378	5.8223	6.3784	7.2077
Critical value	7.8147	7.8147	7.8147	7.8147	7.8147
P-value	0.4290	0.4173	0.1206	0.0946	0.0656
GOF Fitting Rank	First	Second	Third	Fourth	Fifth

From the Ryan-Joiner Test of Normality Assessment (Table 12) and Figure 6 (histogram and normal quantile plot) of the tree circumference data as shown above, it is obvious that the shape of the tree circumference data is skewed to the right and heavy tailed. This is also confirmed from the skewness (2.35) and kurtosis (10.89) of the tree circumference data as computed in Table 11. Since fitting of a probability distribution to the tree circumference data may be helpful in predicting the probability or forecasting the frequency of occurrence of the tree circumference, this suggests that 'y', the tree circumferences, could possibly be modeled by some skewed distributions. As such we have tested the fitting of the Chaudhry and Zubair's GGIGD, gamma and lognormal distributions based on their goodness of fit to the tree circumference data (as given in Table 10). For this, Maple 11 has been used for computing the data moments, estimating the parameters and chi-square test for goodness-of-fit. The data moments are computed as $\hat{\mu}_1 = 4.5350, \hat{\mu}_2 = 27.2425, \hat{\mu}_3 = 221.5905$ and $\hat{\mu}_4 = 2299.2309$. The estimation of the parameters and chi-square goodness-of-fit test are provided in Tables 13 and 14 respectively.

From the chi-square goodness-of-fit test, we observed that the Chaudhry and Zubair (2002)'s models (GGIGD 1, 2 and 3), lognormal and gamma distributions fit the tree circumference data reasonably well. However, the Chaudhry and Zubair [1]'s GGIGD Model 2 produces the highest p-value and smallest test statistic and therefore fitted better than GGIGD Model 1, GGIGD Model 3, lognormal and gamma distributions. Moreover, for the parameters estimated in Table 13, the Chaudhry and Zubair [1]'s models (GGIGD 1, 2 and 3), lognormal and gamma distributions have been superimposed on the histogram of the tree circumference data as shown in Figure 7, from which we observed that the Chaudhry and Zubair [1]'s GGIGD Model 2 fits the tree circumference data reasonably well.

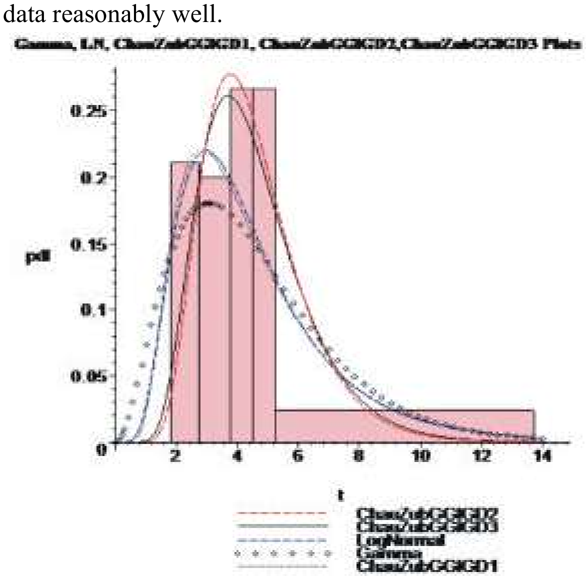


Fig. 7: Fitting of the pdfs of the Chaudhry and Zubair (2002) (GGIGD 1, 2 and 3), Lognormal and Gamma distributions to the Tree Circumference Data

Concluding Remarks: In this paper, we have considered the generalization of the generalized inverse Gaussian distribution (GGIGD) introduced by Chaudhry and Zubair [1], conducted some of its statistical analysis. We have reviewed the GGIGD model first and then established its several new distributional properties, including the reliability analysis, the estimation of the parameters and computations of percentage points. We have used two real life-time data to show the applications of the Chaudhry and Zubair [1]'s GGIGD model. It is hoped that the findings of this paper will be quite useful to the researchers and practitioners in various fields of theoretical and applied sciences.

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