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An Investigation of Confidence Ellipses Parameters using Weibull Distribution

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Abstract: The purpose of this paper is to compare the efficiency of asymptotic joint distribution and the Fisher information matrix, in the construction of confidence ellipses parameters for Weibull distribution. The resulting coverage probabilities are used as the criteria for comparing against the confidence coefficient 0.98. The Monte Carlo simulation method is used to investigate the accuracy of the confidence ellipses. The results showed that the confidence ellipses can be achieved when $\alpha = 2.0$ or greater and the sample size (n) increases. The results also show that the covariance matrix derived from the Fisher information matrix offers greater accuracy of confidence ellipses, at 0.9901 when $\alpha = 5.0$ and sample size (n) = 1,000.

Key words: Confidence ellipses ・ Confidence coefficient ・ Asymptotic joint distribution ・ Fisher information matrix ・ Monte Carlo simulation

INTRODUCTION

The Weibull distribution offers models which are frequently used with regard to lifetime models. It offers a high degree of flexibility, meaning it can result in increasing and decreasing failure rates. This flexibility and the variety of shapes means that it can serve many functions, including but not limited to, insurance, industrial engineering and weather forecasting. It offers two key functions, namely; cumulative distribution function (cdf) and probability density function (pdf). It can be denoted as follows; $W(\alpha, \beta)$, where α and β are the parameters used. Two parameter Weibull distributions can be described as follows:

$$F(x;\alpha,\beta) = 1 - e^{-(x/\alpha)^{\beta}}, x > 0,\ \alpha > 0,\ \beta > 0 \quad (1.1)$$

and

$$f(x;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^{\beta}},\ x > 0 \quad (1.2)$$

Within the examples above, α operates as the scale parameter and β functions as the shape parameter. These models also incorporate exponential distribution, as in decreasing failure rate $\beta < 1$, constant failure rate $\beta = 1$ and increasing failure rate $\beta < 1$.

Waloddi Weibull [1] suggested that to solve the non-linear equation iterative schemes called maximum likelihood estimators (MLEs) can be used. However, they found that those expressions for β did not function correctly. Cohen and Betty [2] suggested modified moment estimators (MMEs) in an attempt to deal with this issue. It is important to note that there is no way to guarantee that the upper bonds of the intervals will always be positive.

Weibull distribution is only one of the popular methods for investigating failure data. Other popular methods are Inverse Gaussian, Gamma, Birnbaum-Saunders and Lognormal. These distributions all work well in the central region of the distribution. However under certain circumstances different distribution methods may be more appropriate. Take for example an engineer seeking to discover the lower percentile of the failure distribution. When the data works well under several distributions, the engineer can then choose the

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distribution which possesses theoretical support. If the engineer is interested in the fatigue model then the Weibull distribution would be an apt choice.

The focus of this paper is on elliptical confidence regions of parameters for the Weibull distribution, i.e. the simultaneous estimation of both parameters. Under certain circumstances, particularly with regard to practical problems more than one statistical interval may need to be considered simultaneously from the same data. To achieve this we need to use simultaneous estimation of both parameters. If one was to estimate the confidence intervals separately at 99 percent, the main issue that would arise is that parameters would not provide 99 percent confidence. The probability of both being correct would only be 0.99^2 or 0.98 (98 percent).

This paper will focus on Weibull distribution. This paper will use two models to determine which model results in confidence ellipses at 98 percent confidence level. The models to be used are asymptotic joint distribution and the Fisher information matrix, Model I and Model II respectfully. Both models will then be investigated for accuracy through the Monte Carlo simulation method. Results showing the actual coverage probability will then be compared with the nominal confidence coefficient, 0.98.

Backgrounds

The Covariance Matrix of Parameter Estimation by Method of Moments and Confidence Region of Parameters for the Weibull distribution (Model I): The $N_2(\theta, \Sigma^{-1})$ distribution assigns probability $1 - \alpha$ to the ellipses

$$\left(\hat{\theta}_n^{(MME)} : \left(\hat{\theta}_n^{(MME)} - \theta\right)' \Sigma^{-1} \left(\hat{\theta}_n^{(MME)} - \theta\right) \le \chi^2_{(2)}(\alpha)\right)$$

where $\chi^2_{(2)}(\alpha)$ denotes the upper 100 α percentile of the $\chi^2_{(2)}$ distribution.

For the first model, E.O. Balitskaya and L.A. Zolutukhina [3] show that the covariance matrix of the estimators for the two parameter Weibull distribution

$$K = \frac{1}{n}\begin{pmatrix} \dfrac{\alpha^2}{\beta^2}\left(1 + \dfrac{\psi^2(2)}{\psi'(1)}\right) & 0 \\ 0 & \dfrac{\beta^2}{\psi'(1)} \end{pmatrix} \tag{2.1}$$

A 100(1-α)% confidence region for parameter $\theta' = (\alpha, \beta)$ of a two dimensional normal distribution is the ellipses determined by all θ such that

$$\left(\hat{\theta}_n^{(MME)} - \theta\right)' \Sigma^{-1} \left(\hat{\theta}_n^{(MME)} - \theta\right) \le \chi^2_{(2)}(\alpha)$$

$$\left(\hat{\theta}_n^{(MME)} - \theta\right)' \Sigma^{-1} \left(\hat{\theta}_n^{(MME)} - \theta\right) \le \chi^2_{(2)}$$

where

$$(\alpha) = \begin{pmatrix} \hat{\alpha}_n^{(MME)} - \alpha & \hat{\beta}_n^{(MME)} - \beta \end{pmatrix} \Sigma^{-1} \begin{pmatrix} \hat{\alpha}_n^{(MME)} - \alpha \\ \hat{\beta}_n^{(MME)} - \beta \end{pmatrix}$$

Model I: The 100(1-α)% confidence region for θ consists of all value (α, β) satisfying

$$\frac{1}{n}\left(\frac{\beta^2}{\alpha^2}\left(1 + \frac{\psi'(1)}{\psi^2(2)}\right)(\hat{\alpha}_n^{(MME)} - \alpha)^2 + \frac{\psi'(1)}{\beta^2}(\hat{\beta}_n^{(MME)} - \beta)^2\right) \le \chi^2_{(2)}(\alpha) \tag{2.2}$$

The Fisher Information of Parameters for the Weibull Distribution (Model II): For the second model, Paul Larsen [4] showed that the Fisher information matrix of θ, where $\theta = (\alpha, \beta)$ is a two dimensional vector of parameter denoted by $\mathrm{I}(\theta)$ defined as

$$\mathrm{I}(\theta) = \mathrm{I}(\alpha, \beta) = \begin{pmatrix} \dfrac{1}{\alpha^2}\left(\psi'(1) + \psi^2(2)\right) & 0 \\ 0 & \dfrac{\alpha^2}{\beta^2} \end{pmatrix} \tag{2.3}$$

A 100(1-α)% confidence region for parameter $\theta' = (\alpha, \beta)$ of a two dimensional normal distribution is the ellipses determined by all θ such that

From (2.1) $\left(\hat{\theta}_n^{(MME)} - \theta\right)' \Lambda^{-1} \left(\hat{\theta}_n^{(MME)} - \theta\right) \le \chi^2_{(2)}(\alpha)$

$$\left(\hat{\theta}_n^{(MME)} - \theta\right)' \mathrm{I}_n(\theta) \left(\hat{\theta}_n^{(MME)} - \theta\right) \le \chi^2_{(2)}(\alpha)$$

Which is equivalent

$$\left(\hat{\theta}_n^{(MME)} - \theta\right)' n\mathrm{I}_n(\theta) \left(\hat{\theta}_n^{(MME)} - \theta\right) \le \chi^2_{(2)}(\alpha)$$

Model II: The 100(1-α)% confidence region for θ consists of all value (α, β) satisfying

$$\left(\alpha^2\left(\psi'(1)+\psi^2(2)\right)(\hat{\alpha}_n^{(MME)}-\alpha)^2+\frac{\beta^2}{\alpha^2}(\hat{\beta}_n^{(MME)}-\beta)^2\right)\leq\chi^2_{(2)}(\alpha) \qquad (2.4)$$

Monte Carlo Simulation Results: In order to compare the efficiency of all confidence regions, a simulation study for different sample sizes and for different parameters values was performed. Data used was as follows: the sample size as n = 10, 100 and 1,000 and the shape parameter as $\alpha = 0.1, 0.5, 1.0, 2.0$ and 0.5. Since β is the scale parameter, β was kept fixed at 1.0, without loss of any generality. The experimental data are generated by the simulation technique using R program version 3.5.0. For each situation, the experiment is repeated 10,000 times to obtain the coverage probability. The results so obtained are reported in Table 1.

The 98% confidence regions for α and β based on the method of moment estimators are given by

Model I: $\frac{1}{n}\left(\frac{\beta^2}{\alpha^2}\left(1+\frac{\psi'(1)}{\psi^2(2)}\right)(\hat{\alpha}_n^{(MME)}-\alpha)^2+\frac{\psi'(1)}{\beta^2}(\hat{\beta}_n^{(MME)}-\beta)^2\right)\leq\chi^2_{(2)}(\alpha)$

Model II: $\left(\alpha^2\left(\psi'(1)+\psi^2(2)\right)(\hat{\alpha}_n^{(MME)}-\alpha)^2+\frac{\beta^2}{\alpha^2}(\hat{\beta}_n^{(MME)}-\beta)^2\right)\leq\chi^2_{(2)}(\alpha)$

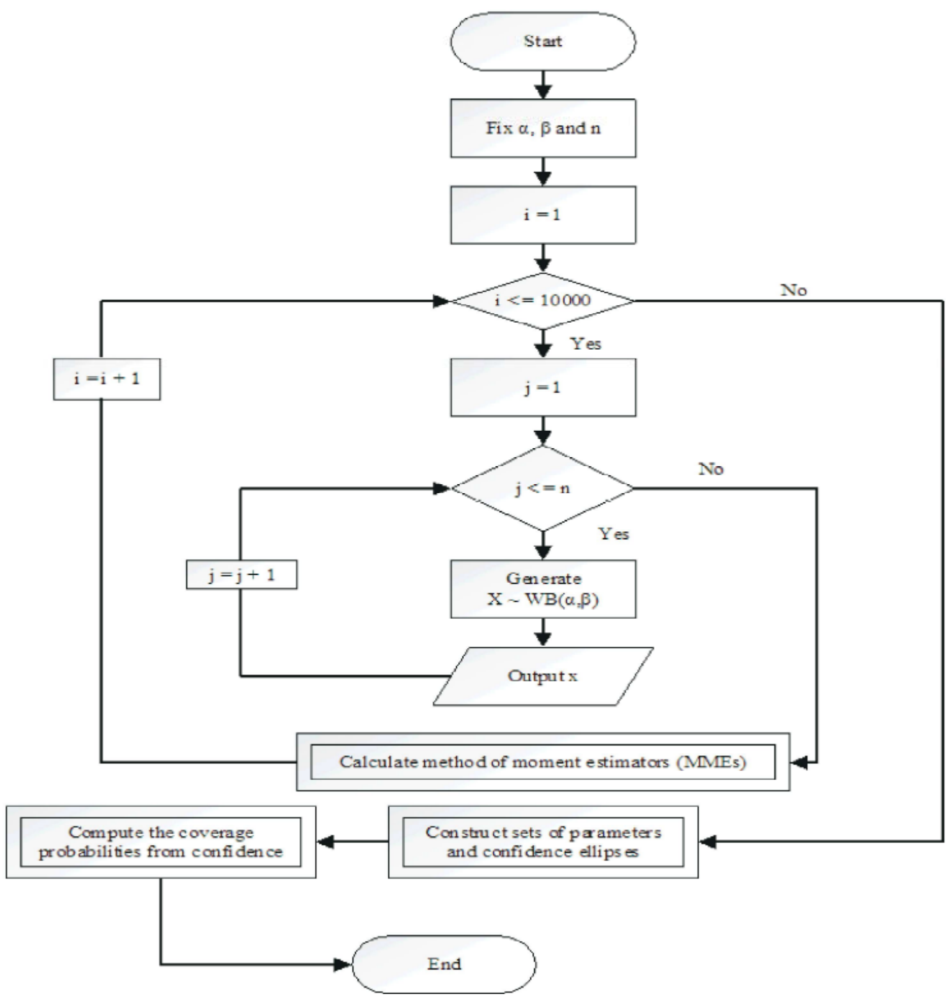

Fig. 1: Programming Flowchat

Table 1: Method of Moment estimates of α and β, their errors and coverage probabilities for confidence ellipses at 98% confidence level (Model I)

			Method of Moment estimates		The percentages of Absolute relative errors		
n	α	β	$\bar{\hat{\alpha}}_n(MME)$	$\bar{\hat{\beta}}_n(MME)$	$\bar{\hat{\alpha}}_n(MME)$	$\bar{\hat{\beta}}_n(MME)$	Coverage probabilities
10	0.1	1.0	1.0043	1.0057	915.580	0.362	0
	0.5	1.0	1.1355	1.0163	118.342	3.498	0
	1.0	1.0	1.4712	1.0581	40.165	5.130	0
	2.0	1.0	2.1857	1.1471	6.203	13.975	0.0017
	5.0	1.0	5.0233	1.3294	8.595	28.852	0.9364
100	0.1	1.0	1.0056	1.0026	915.671	0.007	0
	0.5	1.0	1.1465	1.0102	120.501	0.952	0
	1.0	1.0	1.5321	1.0271	43.270	0.681	0
	2.0	1.0	2.3609	1.0856	12.582	0.920	0.8843
	5.0	1.0	5.1278	1.1713	1.682	1.775	0.9651
1,000	0.1	1.0	1.0061	1.0004	915.753	0.003	0
	0.5	1.0	1.1472	1.0034	120.624	0.006	0
	1.0	1.0	1.5368	1.0153	44.824	0.059	0
	2.0	1.0	2.3741	1.0342	12.463	0.146	0.8548
	5.0	1.0	5.1430	1.0139	1.997	0.398	0.9892

Table 2: Method of Moment estimates of α and β, their errors and coverage probabilities for confidence ellipses at 98% confidence level (Model II)

			Method of Moment estimates		The percentages of Absolute relative errors		
n	α	β	$\bar{\hat{\alpha}}_n(MME)$	$\bar{\hat{\beta}}_n(MME)$	$\bar{\hat{\alpha}}_n(MME)$	$\bar{\hat{\beta}}_n(MME)$	Coverage probabilities
10	0.1	1.0	1.0040	1.0050	915.612	0.068	0
	0.5	1.0	1.1272	1.0172	118.241	1.382	0
	1.0	1.0	1.4412	1.0405	53.212	3.658	0
	2.0	1.0	2.1984	1.1420	6.403	13.692	0.0023
	5.0	1.0	5.1632	1.2256	8.955	27.103	0.9702
100	0.1	1.0	1.0049	1.0036	915.870	0.005	0
	0.5	1.0	1.1378	1.0104	120.586	0.179	0
	1.0	1.0	1.5621	1.0365	43.812	4.982	0
	2.0	1.0	2.3560	1.1307	11.568	0.996	0.8972
	5.0	1.0	5.5814	1.2043	2.293	1.183	0.9893
1,000	0.1	1.0	1.0058	1.0015	915.895	0.004	0
	0.5	1.0	1.1395	1.0073	120.769	0.003	0
	1.0	1.0	1.5740	1.0318	40.337	0.095	0
	2.0	1.0	2.4914	1.1149	11.820	0.072	0.9213
	5.0	1.0	5.6431	1.1975	2.581	0.298	0.9901

Table 1 shows that the coverage probabilities of confidence ellipses of parameters the Weibull distribution, using asymptotic joint distribution, cannot work when α is less than 2.0. It can also be derived that $\alpha = 2.0$ cannot work with a sample size (n) of 10, in fact $\alpha = 2.0$ only offers a workable solution when sample size (n) is 1,000. The greatest results are achieved with a sample size (n) of 1,000 and when $\alpha = 5.0$, showing a result of 0.9892 respectfully.

Table 2 displays the results of the coverage probabilities of confidence ellipses of parameters the Weibull distribution, using the Fisher information matrix. The results show that again $\alpha = 2.0$ only works with a sample size (n) of 1,000. A lower α value or sample size (n) does not provide a workable solution. When $\alpha = 2.0$ and sample size (n) = 1,000 the result is a respectful 0.9213 but this still falls short of the confidence coefficient 0.98 goal. An α value of 5.0 achieves the result with a sample size (n) of both 100 and 1,000 respectively.

Tables 1 and 2 show us; that the coverage probability of confidence ellipses for parameters of Weibull distribution is intrinsically linked to both sample size and α, when β is fixed at 1.0. Firstly in relation to α; both tables show that an α value of less than 2.0 produces results of 0. Thus, the confidence ellipse that we construct cannot cover that set of parameter estimates that are constructed. Therefore to achieve coverage probability of confidence coefficient at 0.98, α must be greater than or equal to 2.0. Using asymptotic joint distribution an α value of 2.0 does not achieve the desired result at any of the tested sample sizes (n), in fact $\alpha = 5.0$ only achieves the objective when the sample size (n) = 1,000; showing a result of 0.9892. The Fisher information matrix offers more usable solutions. It shows that when $\alpha = 2.0$ the confidence coefficient at 0.98 target can be achieved with a sample size (n) of 1,000. The target may also be achieved when $\alpha = 5.0$ with a sample size (n) of 100 (0.9893) and a sample size (n) of 1,000 (0.9901) respectfully.

Concluding Remarks: The above investigation shows that in Model I and Model II; as α values increase and sample size (n) increases, then so do confidence ellipses. Model I produced a coverage probability of 0.9892 with $\alpha = 5.0$ and sample size (n) = 1,000. This is the highest result produced by Model I. Model II produced a high of 0.9901 with $\alpha = 5.0$ and sample size (n) = 1,000.

These results show that in the Weibull distribution we can utilize MMEs (method of moment estimators) when using the covariance matrix from the Fisher information matrix as opposed to the covariance matrix from the asymptotic joint distribution for the construction of confidence ellipses due to the higher efficiency of the coverage probabilities.

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