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Tracing the Shadow: Mathematical Calculation of Prayer Times Using Spherical Trigonometry

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Abstract: In this work, complex but not complicated spherical trigonometry functions were used to determine the Islamic prayer times. To the best of our knowledge, it is the first time that a mathematical modelling of the relative shadow height of objects at zenith time has been visually presented for each Julian day of the year in great detail; this is a requirement step for the end and beginning of Zuhr and Asr prayer times, respectively. For practical and proof-of-concept purpose – albeit its applicability to all locations without exception, the prayer times for the capital cities of the under-investigated region of East African countries i.e. Kenya, Somalia, Ethiopia and Djibouti were chosen as case studies for mathematical calculations and data interpretation in addition to Mecca as a benchmark. All the mathematical equations needed for the determination of the beginning for each prayer time were thoroughly reviewed. For data analysis, the widely available MS Excel programme was utilized. The inter-correlation of the shape and the fringe pattern of prayer time variations with the days of the year was extensively elucidated and analyzed in relation to the geographical coordinates as well as the Sun declination and elevation/depression angles associated with the particular locations studied. Interestingly, it has been exhibited that the curvature of prayer times for Mecca has a unique and distinctive shape due to the unique positioning of Mecca. The hour angles, Sun elevation/depression angles for tracing the temporal changes in the length of object's shadow and the onset for the evening and/or the true morning twilights as a function of Julian day number were discussed in great detail.For the determination of the start of Isha'a and Fajr prayer times, we adopted angle 18° for its wide acceptance in the region. All the cities chosen fall in close proximity and in the northern hemisphere except Nairobi which occurs just below the equator in the southern hemisphere. Moreover, all these cities including Mecca are in the same time zone (UTC + 3).

Key words: Prayer time · Spherical trigonometry · Shadow length/height · Elevation angle · Depression angle · Hour angle

INTRODUCTION

The determination of the strictly specified prayer times in Islam is an imperative and a prerequisite condition for performing obligatory prayers. There are five daily obligatory prayers that should be observed in Islam which coincide with significant temporal changes of the Sun's position as the earth rotates on its celestial axis and moves through its various stations during its elliptical revolution around the Sun.

In the Quran[1] and the teachings of Prophet Mohammed[2] (Peace and Blessing of Allah be upon him), a beginning and an end time limit is exclusively assigned to each prayer and in most of the prayers the end of one prayer time marks the beginning of another. There are no gaps – for instance - between the end and the beginning times of Zuhr, Asr, Maghrib and Isha'a prayers; in most Islamic jurisprudential schools - though not undisputed, Isha' time extends until the beginning of Fajr. The only unquestionable gap between two consecutive prayers occurs between the end of Fajr at the sunrise and the beginning of Zuhr prayer at the decline of the Sun from the zenith of its local meridian.

[1]Qur'an: Al-nisaa (Chapter 4), verse: 103 "Prayer indeed has been enjoined on the believers at fixed times".
[2]Hadith Jibril narrated by Ahmed, Nisa'i and Tirmidi.

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Performing an obligatory prayer outside its respectively prescribed time intervals is strictly forbidden and makes particular prayers invalid. Therefore, determining the exact times that prayers could or could not be performed is as important as the prayers themselves in a muslim person's life.

The empirical verification of prayer times by simple observation and inspection is quite easy in parts of the world where the day and night are relatively balanced and/or there are slight changes in day/night hours in the various seasons of the year. However, in other parts of the world (high locations), observing the sun movement and thus the accurate observation of prayer times empirically proves enormously difficult due to the extreme imbalance of day and night lengths, cloudiness of the sky, brightness of illuminations in urban areas as well as the sun's position and declination.

Moreover, the use of technology and mathematical modeling in determining prayer times in high locations is crucial if not imperative. In lower or medium-range (locations where sun movement is easily verified and observed by ordinary individuals), it would – to a large extent - be very helpful guide in time keeping and advance planning. Nevertheless, it is noteworthy that the Islamic Sharia obligatory worships and duties (prayers, Hajj or Ramadan) are based on simple naked-eye observations and sightings by ordinary people – in this respect, the knowledge of technology and advanced mathematics is not necessary though it might help[3]. If there is a clear disparity in the correct prayer timing between ordinary observations (rough estimates) and the timings determined using mathematics and technology, the ordinary observations should be given the precedence and used as a baseline in correcting any model employed in mathematical calculations; simplicity and easiness is one the basic tenets of Islamic Sharia.

In Islamic heritage and medieval civilizations, devices such as astrolabes were utilized to decipher the correlation between the "*apparent movements"* of the sun (and other celestial bodies) monitored from a particular latitude on earth and "*time"* using primitive but otherwise sophisticated instruments and mathematical precision [1-6]. David A King noted that "The Arabs of the peninsula in the time before Islam possessed an intimate knowledge of the apparent motion of the sun, moon and stars across the heavens, the months and seasons, the changing night sky throughout the year and the associated meteorological phenomena" [7]. In this era of scientific and mathematical advancement, prayer times can easily be calculated using computer modeling and spherical trigonometrical functions. Nowadays, many people rely upon internet and smart phone applications for prayer timetables though the reliability of these sources are questionable especially the timing of Isha'a and Fajr prayer where there is a great degree of variations in the methods used among different Islamic countries, organizations and scholars.

In this article, we aim to achieve the following objectives: i) to use mathematical spherical trigonometry functions with the help of computer software programmes such advanced MS Excel and mathematical packages such as Maple to calculate all prayer times; ii) for some prayers such as Asr (the afternoon prayer), the correct timing depends on the length of the shadow of objects and for the first time - to the best of our knowledge –we introduce an elaborate visual representation that traces the temporal and irregular periodic changes in the relative zenith shadow length owing to the variations of geographical coordinates and Sun declination at noon time. The model depicts graphically the non-monotonic relationship between the zenith length and the Julian day number (JDN). The significance of this representation is exhibited in the determination of the exact ratio of the length of zenith shadow of any object compared to the height of the object throughout the year.

For practical applications, we chose to use the coordinates of the under-investigated geographical region of East African countries (the capital cities of Somalia, Mogadishu; the capital city of Kenya, Nairobi and the capital city of Ethiopia, Addis Ababa) and as a benchmark the prayer times of Mecca was also used for comparison purpose.

Theory and Methodology

An Overview: In this section, the mathematical formulas and methods employed to calculate prayer times will be discussed in depth.

Besides, we first define the terms for the five prayer times as follows:

- *Zuhr (the noon) prayer time*: commences immediately after midday when the Sun has passed the local meridian (reaching the highest point in the sky) of a particular location and ends when the shadow of an object is equal to its height

[3]The hadith: "We are an unlettered nation; we don't write or calculate. The month is such and such or such and such meaning that it is sometimes twenty nine and sometimes thirty" - narrated by Muslim in his authentic book.

discounting the object's shadow at zenith time (the shadow at midday). There always exists a zenith shadow throughout the year except just two times at places located between the two tropics (the tropic of Cancer and the tropic of Capricorn).

- *Asr (the afternoon) prayer time*: commences when the Zuhr time terminates and ends when the Sun sets. In Hanafi school of jurisprudence, the Zuhr ends and Asr begins when the shadow of an object is twice its height plus any shadow that remains at zenith time.
- *Maghrib (sunset) prayer time*: commences when the Sun disappears below the horizon at sunset (dusk time).
- *Isha'a (evening) prayer time:* commences at the disappearance of red[4] and/or the diffuse white twilight (due to the Rayleigh light scattering of the Sun) after the sunset when the darkness of the night falls and there is no light scattering left in the sky. The Isha'a time extends – on the other hand - until midnight[6] or true morning twilight (Fajr time) based on differing opinions of the most common and the majority of well-established Islamic jurisprudence schools.
- *Fajr (dawn) prayer time:* commences at the true morning twilight when the sky begins to lighten up at dawn time and the light scattering spreads laterally on the horizon. Fajr prayer time ends at sunrise.

The Significance of Spherical Trigonometry Functions: The basics of the subject of spherical trigonometry was dealt with by early Greek mathematicians and further advanced by the prominent scholars of Islamic Caliphates that existed in the Middle East, North Africa and Spain during 8th and 14th centuries [3, 6, 8] to determine the direction of the Qibla. Remarkably, in his great book of "*The Book of Unknown Arcs of a Sphere*", Abu Abdallah Al-Jayyani has formulated the solutions of a spherical triangle and sine laws [9] which later had a strong influence on European mathematics [6].

Spherical trigonometry deals with triangles drawn on a sphere (Fig. 1a) and its development led to significant improvements in the arts of navigation, geographic map-making, the position of sunrise/sunset and astronomy [8]. With the use of a set of spherical cosine and sine rules (Eqs. 1-4), the sides ("a", "b" & "c") and the angles of ABC triangle (Fig. 1a) can be measured straightforward application determining the position of the Sun or any other celestial bodies observed from any particular location on earth and/or the navigation between two points on the surface of the earth [10, 11]:

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A) \quad (1)$$

$$\cos(b) = \cos(a)\cos(c) + \sin(a)\sin(c)\cos(B) \quad (2)$$

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(c)\cos(C) \quad (3)$$

$$\frac{\sin(a)}{\sin(A)} = \frac{\sin(b)}{\sin(B)} = \frac{\sin(c)}{\sin(C)} \quad (4)$$

The more familiar cosine and sine rules of plane triangle is the limiting case of the spherical triangle; the derivation of these equations are beyond the scope of
[5] this current article. Nonetheless, the spherical triangle differs from the plane triangle in that the sum of the angles is more than 180°.

The Azimuth and Elevation Angles: A similar triangle to the one shown in Fig. 1a - governed by the same spherical trigonometric functions - can be used to depict the position of the Sun with respect to the observer's position at any location on earth as illustrated in Fig. 1b. Given the geographical coordinates (longitude and latitude) and the Sun's declination angle, δ, (the angle the Sun makes with the equatorial plane at any instant of the earth), the position of any point of earth's surface with respect to the Sun or any other point/location can be known [12, 13]. Conversely, the azimuth and the elevation angles, as shown in Fig. 1 (sides "a" and "c", respectively in Fig. 1a and as labeled in Fig. 1b, correspondingly), define the position of the Sun (or any other celestial body) in the sky as observed from a particular location at any time.

The word "azimuth" has been taken from the Arabic word "Assamt" meaning "direction" and the azimuth angle defines the compass direction (aka compass bearing) of the Sun along the horizon relative to the true North Pole (Fig. 1b). The azimuth angle denotes the angular horizontal distance measured clockwise that the Sun's vertical axis makes with the North Pole as the Sun position changes during the day from sunrise to sunset. At equinoxes, the Sun rises directly from the east and sets directly at the west making azimuth angles with reference to the North Pole of 90° and 270°, respectively. The elevation angle, α, (aka the "altitude" angle), on the other hand, measures the angular height of Sun in the sky measured from the horizontal plane.

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[4]The majority of jurisprudence schools except Hanafi School.

[5]Hanafi School.

[6]Hanbalia School.

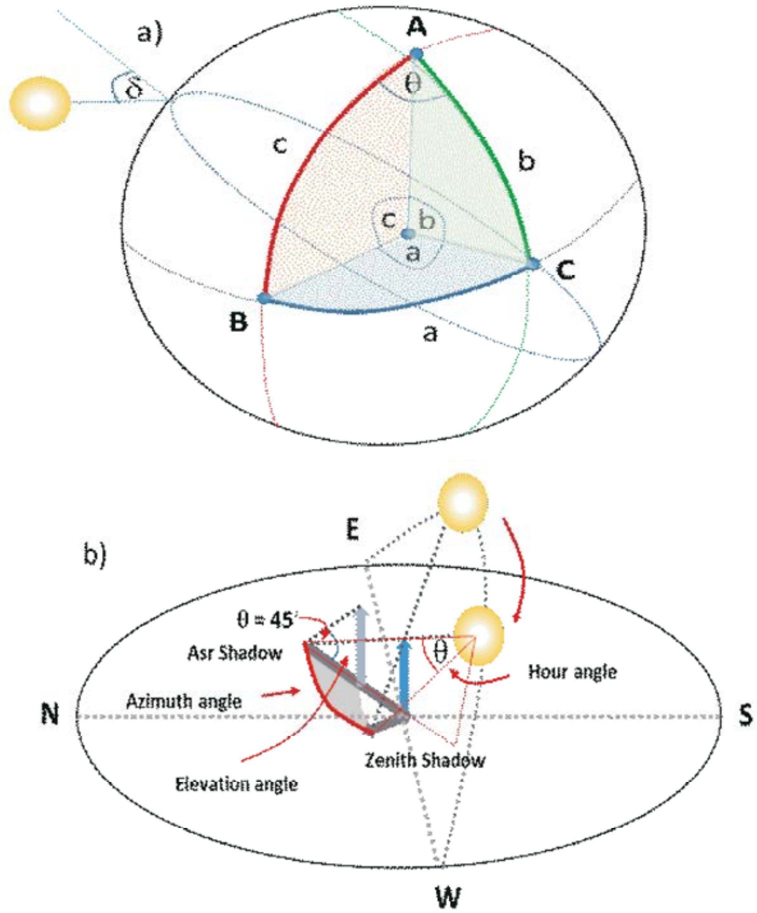

Fig. 1: Schematic representation of great circles forming a triangle on a sphere and its application on shadow movements; a) three great circles on a sphere forming ABC triangle and b) shadow movement between Zuhr and Asr time

Besides, the elevation angle, α, varies throughout the day and its maximum value at zenith time depends on the latitude, φ, of particular locations and the Sun's declination angle, δ, of the corresponding day. The maximum elevation angle, α , of any location in any day of the year can be found as:

$$\alpha_M = 90 \pm \varphi - \delta \quad (5)$$

where α_M is the maximum elevation angle; φ is the latitude of the location of interest and has a positive value for the northern hemisphere and negative value for the southern hemisphere; δ is the Sun declination angle which depends on the day of the year. At winter and spring equinoxes, the elevation angle, α, is 0° at astronomical sunrise and 90° when the Sun is directly overhead (at zenith time). If Eq. 5 gives greater than 90°, the results should be subtracted from 180° which means that the Sun at noon time is coming from the south as is typical in the northern hemisphere.

The closely related angle with the elevation angle is the zenith angle, ς, which has the same meaning but measured from the vertical axis (the overhead angle of vertical and upright objects) of the Sun rather than the horizontal plane and is equal to:

$$\varsigma = 90° - \alpha \quad (6)$$

The Sun declination angle, δ, depends on the Julian number of each day of the year and can – with very good approximation – be obtained from Eq. 7 [14]:

$$\delta = -23.45° \times \cos\left[\frac{2\pi}{365} \times (d + 10)\right] \quad (7)$$

where d is the Julian day number (JDN). The only independent variable in Eq. 7 is the day of the year. Plotting this equation using MS Excel gives Fig. 2. At equinoxes (March 21st and Sept 21st), the Sun declination at the equator is zero as shown in Fig. 2. At the winter and summer solstices (Dec 31st and Jun 21st),

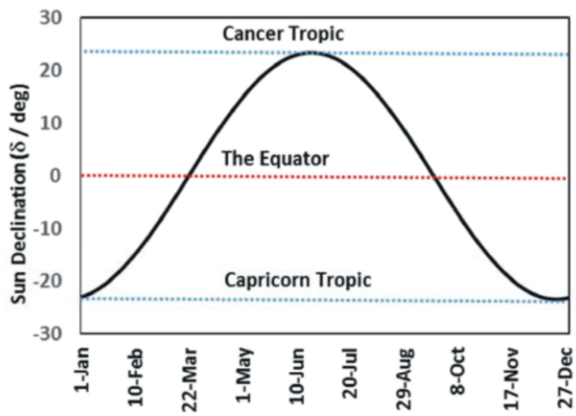

Fig. 2: Sun declination angle as a function of Julian day number throughout the year showing zero declinations at equinoxes and going through minimum and maximum declinations at the solstices with respect to the equator. For presentational purposes, the Julian day numbers are converted to commonly known Gregorian days and months.

the Sun declination with respect to the equator is ±23.45° reaching its minimum and maximum angles (+ve at the tropic of Cancer and –ve at the tropic of Capricorn) due to the tilt of the earth on its axis of rotation.

The Hour Angle: The angle marked as "θ" Fig. 1a and/or the schematic representation of the Sun position in Fig. 1b denotes the "*hour angle"* which is an expression describing the time difference between the apparent local solar time (Asr time, for example) and solar noon time (i.e. zenith time). This angle is associated with the changes that the azimuth and elevation angles undergo during the day. Therefore, the complete equation for the elevation angle, α, at any time of the day can be obtained – based on the spherical cosine rules shown above - as follows:

$$\alpha = \sin^{-1}[\sin(\delta)\sin(\varphi) + \cos(\delta)\cos(\varphi)\cos(\theta) \qquad (8)$$

where α, δ, φ have their corresponding meanings as defined above and θ is the hour angle associated with the sequential variations of the elevation angle during the day. Conventionally, the hour angle, θ, is 0° at zenith time when the Sun elevation angle is maximum, α_M and Eq. 8 reduces to:

$$\alpha - \sin^{-1}[\sin(\delta)\sin(\varphi) + \cos(\delta)\cos(\varphi) \qquad (9)$$

Using the trigonometric identity "cos(δ –φ)" in Eq. 9, the limiting case of the maximum elevation angle, α_M, is found – Eq. 10 is exactly the same as Eq. 5 (above):

$$\alpha_M = \sin^{-1}[\cos(\delta - \varphi)] = \sin^{-1}[\sin(90° - [\delta - \varphi])] = 90° + \varphi - \delta \qquad (10)$$

With a simple algebraic rearrangement of Eq. 8, the equation of the hour angle, θ, for any elevation angle can found as:

$$\theta = \cos^{-1}\left[\frac{\sin(\alpha) - \sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] \qquad (11)$$

At astronomical sunrise with α = 0°, the hour angle, $\theta_{sunrise}$, is equal to:

$$\theta\text{sunrise} = \cos^{-1}\left[\frac{-\sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] = \cos^{-1}[-\tan(\delta)\tan(\varphi)] \qquad (12)$$

The hour angle, θ, is the parameter of greatest interest in this work as it enables us to determine the correct time it takes the Sun to reach the elevation angle, α, corresponding to the length of the shadows of objects at any time of the day. This iskey for the calculations of prayer times as will be discussed below in section 2.6.

The Equation of Time (EoT): The earth's motion around the Sun is not perfectly a uniform circle due to the earth's axial tilt of 23.45° relative to the Sun's plane and the elliptic path it takes around the Sun, i.e. the earth does not travel around the Sun at a constant speed throughout the year but faster when it is nearer (at perihelion) than when it is farther away (at aphelion) according to the Kepler's laws of planetary motion. The earth also spins at an irregular rate around its axis of rotation. This irregular motion and spinning of the earth creates a time discrepancy between the apparent solar time tracking the diurnal motion of the Sun and the mean solar time (clock time) tracking a theoretical mean of the solar day (24 hour day with no change throughout the year). The apparent solar time– for instance - can be behind the mean solar time, i.e. clock timeby ~ 15 min in February and ahead by ~ 15 min in November each year so day lengths may vary by up to 30 min throughout the year.

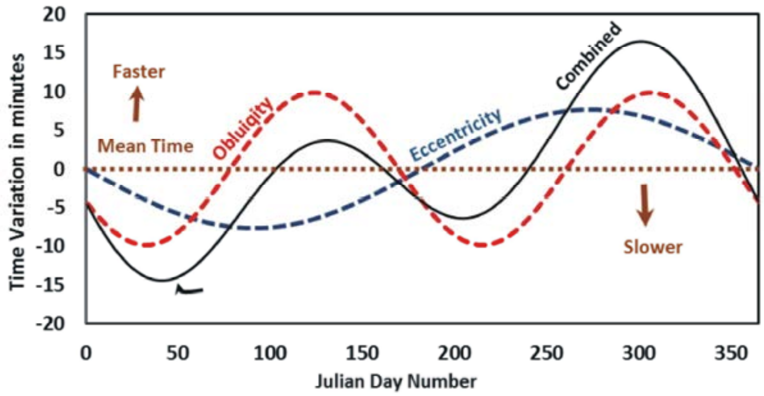

Fig. 3: Shows the calculated results of the Equation of Time (EoT) as the function of Julian day number of the year. The red dotted curve represents the contribution of the *ecliptic obliquity* factor (amplitude ~ 9.87 and half a year period, i.e. two cycles a year) and the blue dotted line denotes the contribution of *orbit eccentricity* factor (amplitude ~ 7.66 and one year period, i.e. one cycle) into the time variation between the apparent solar time and mean solar time. The solid black curve shows the combined effects of the two factors expressed as the Equation of Time

The periodicity of the time variation between the apparent solar and mean solar times is described in the "*The Equation of Time (Eo*T)" [15] composed of two periodic functions resulting from: the *effect of orbit eccentricity* due to the elliptic path of the earth's revolution around the Sun and the *effect of ecliptic obliquity* due to the tilt of earth's rotational axis as expressed in Eq. 13.

$$EoT = 9.873 \times \sin\left(\frac{4\pi d}{375} + 3.588\right) - 7.655 \times \sin\left(\frac{2\pi d}{365}\right) \quad (13)$$

As defined in Eq. 13 and illustratedin Fig. 3, the effect of *orbit eccentricity* contributes to the Equation of Time (EoT) by a "*sin wave"* variation of time with an amplitude of ca. 7.66 min (second term of Eq.13) and a period of one year, i.e. one cycle. The zero points of this wave function (Fig. 3, the blue dotted line) are reached at perihelion (at the beginning of January when the sun is closest to the earth) and aphelion (beginning of July when the sun is farthest from the sun); the two extreme minimum and maximum points are reached in early April and early October, respectively. Similarly, *the ecliptic obliquity* factor contributes to the Equation of Time (EoT) by a sin wave variation of time with an amplitude of ca. 9.87 min (the first term of Eq. 13) and a period of half a year, i.e. two cycles.The zero points of the ecliptic obliquity wave function are reached at equinoxes and solstices (Fig. 3, the red dotted line) while the extreme minima and maxima points occur at early February and August (minima) and early May and November (maxima). The Equation of Time is, thus, an important baseline correction factor which is a key in determining Sun's position in relation to particular locations at specific times in any day of the year such as the local zenith time (important for Zuhr prayer time).

Equations for Prayer Times: The prayer times for each location is different and the knowledge of the geographical coordinates of particular locations are prerequisite for finding the actual prayer times. The values of Sun declination as well as the correction factor resulting from the "Equation of Time" owing to the time discrepancy between the apparent solar time and mean solar times are universal and depend on the Julian day number (Eqs. 7 & 13).

Zuhr Prayer Time: The Zuhr prayer time can be determinedwhen the Sun passes over its local meridian and reachesits highest elevation angle (the zenith point) i.e. $\alpha = 90°$ which occurs at the apparent solar time of 12:00 pm at equatorial equinoxes with zero latitude angle, $\varphi = 0°$and zero declination angle, $\delta = 0°$. To find out the local zenith time of a particular location (i.e. the apparent local zenith time), three correction factors are applied; i) the standard time zone of particular locations (UTC + 3 in this case) ii) the time difference due to local meridian coordinate and iii) the time variation between the apparent solar and mean solar times described in the "Equation of

Time" (section 2.5, Eq.13) for each day of the year. Therefore, the Equation for Zuhr prayer can be generalized as follows [16-18]:

$$Zuhr(Z) = 12:00 + T - \frac{\lambda}{15} - \frac{EoT}{60} + \tau \quad (14)$$

where T stands for the time zone due to the difference between the local time of particular locations and Prime Meridian, λ is the value of longitude coordinate of specific locations measured in degrees and EoT is the correction factor due to the Equation of Time to which the time variation of Zuhr time from day to day is attributed. The factors "15" and "60" are conversion factors from degrees to hours (1 hr = 15°) and from minutes to hours (EoT gives values in minutes), respectively. The last term in Eq. 14 indicates the short time that it takes the Sun to cross the zenith point which is ~ 4-5 min.

Asr Prayer Time: The beginning of Asr prayer time depends on the height of objects' shadow after the Sun declines to the west. As discussed in section 2.1, there is a consensus among the majority of Islamic jurisprudence schools, that the time of Asr prayer commences when the shadow length is equal to the height of the object (except the Hanafi school). This concept is based on the time when there is very little or no zenith shadow at equinoxes. However, when there is a zenith shadow at noon time (all days for all locations except at equinoxes), the length of this shadow should also be accounted for in determining the beginning of Asr prayer time.

Hence, to calculate the Asr prayer, three parameters should be considered; (a) the zenith shadow at noon time, (b) the Sun elevation angle, α_A, (or more importantly, the zenith angle, ς_A) when the shadow of objects is equal to their height plus the zenith shadow and (c) the hour angle, θ_A, defining the time offset for the height of shadows to reach objects' length (see the schematic representation of shadow movement in Fig. 1b).

The Zenith Shadow, l_Z

The length of zenith shadow, l_Z, at noon time can be obtained from the combination of equations Eq. 5 & Eq. 6 (or Eq. 5 & Eq. 10) when the Sun elevation angle is the maximum and the zenith angle, ς_N, at noon time equals to:

$$\varsigma_N = 90° - \alpha_M = \delta - \varphi \quad (15)$$

The length of the zenith shadow, l_Z can then be calculated from:

$$l_Z = h \times \tan(\varsigma_N) = h \times \tan(|\delta - \varphi|) \quad (16)$$

Since the direction of the shadow is not of interest to us, the zenith angle is kept positive and its magnitude is only considered. When the shadow, l and the object, h, lengths are of equal height, both the elevation, α and the zenith, ς, angles have a value of 45° (Fig. 1b). As a result, the total shadow length, l_A, at the beginning of Asr time including the zenith shadow, l_Z, is equal to:

$$l_A = l + h \times \tan(\varsigma_N) = h \times \tan(45°) + h \times \tan(|\delta - \varphi|) \quad (17)$$

Since tan(45°) = 1, Eq. 17 becomes:

$$l_A = h + h \times \tan(\varsigma_N) = h[1 + \tan(|\delta - \varphi|)] \quad (18)$$

Zenith Angle at the beginning of Asr Time

We can now determine the zenith angle, ς_A, for the beginning of Asr Prayer in any day of the year from the tangential function of the Asr shadow and the height of objects – note the cancelation of "h":

$$\tan(\varsigma_A) = 1 + \tan(|\delta - \varphi|) \quad (19)$$

The arbitrary value of the Asr zenith angle in any day of the year for any location is found from:

$$\varsigma_A = \tan^{-1}[1 + \tan(|\delta - \varphi|)] \quad (20)$$

Given the fact that: $\tan(90° - x) = \cot(x)$, the Asr elevation angle, α_A, is found from:

$$\alpha_A = \cot^{-1}[1 + \tan(|\delta - \varphi|)] \quad (21)$$

The Hour Angle, θ_A, for the beginning of Asr Time

The hour angle, θ_A, for the time offset of the Asr time from zenith point is calculated from the combination of Eq. 11 and Eq. 21 as follows:

$$\theta_A = \cos^{-1}\left[\frac{\sin(\alpha_A) - \sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] \quad (22)$$

Or

$$\theta_A = \cos^{-1}\left[\frac{\sin\left(\cot^{-1}[1 + \tan(|\delta - \varphi|)]\right) - \sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] \quad (23)$$

Table 1.0: Shows the various depression angles used by different institutions for Isha'a and Fajr prayer items in the Islamic World

	Depression Angle (decrees)	
Institution	Fajr	Isha'a
Muslim World League	18	17
Islamic Society of Northern America (ISNA)	15	15
Egyptian General Authority of Survey	19.5	17.5
Umm Al-Qura University, Mecca	18.5	90 min after Maghrib or 120 min after Maghrib for Ramadan
University of Islamic Science, Karachi	18	18
Institute of Geophysics, University of Tehran	17.5	Not explicitly defined
Shi'a Ithna-Asharia, Leva Research Institute, Qum	16	14

Finally, the beginning of Asr prayer time can be obtained by the addition of Zuhr time (Eq. 14) and Asr time offset found from its associated hour angle described in Eq. 23:

$$Asr(A) = Zuhr(Z) + \frac{1}{15}\theta_A = Z + \frac{1}{15}\cos^{-1}\left[\frac{\sin(\alpha_A) - \sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] \quad (24)$$

where α_A is as defined in Eq.21 and the factor "15" is as explained above. For Hanafi School of Jurisprudence, "2" is used instead of "1" in Eq.23 due to the fact that the Asr time begins - according to this school - when the length of the shadow is twice that of the object.

Maghrib Prayer & Sunrise Times: The astronomical sunrise and sunset times are found from the offset times (-ve for the sunrise and +ve for the sunset) with respect to Zuhr time described in the hour angle, θ, when the Sun elevation, α, is 0° degrees as expressed in Eqs. 11 & 12. However, the actual sunset (Maghrib time) or sunrise (the end of Fajr time) does not occur at α = 0° due to the refraction of Sun light by the lower terrestrial atmosphere.

Furthermore, the true or actual sunrise appears slightly before the astronomical sunrise and the actual sunset (Maghrib time) occurs slightly after the astronomical sunset. A combined depression angle of α = -0.833° associated with the horizontal refraction angle from the Almanac and the semi-diameter of the Sun is generally accepted for the actual sunset/sunrise [16-18]. Therefore, the equation for the actual sunset, i.e. Maghrib time is – based on the fixed angle of α = - 0.833° hour angle offset time as follows:

$$Maghrib\ (M) = Z + \frac{1}{15}\cos^{-1}\left[\frac{\sin(-0.833°) - \sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] \quad (25)$$

Similarly, the equation for the actual sunrise is defined as (note the –ve sign):

$$Sunrise = Z - \frac{1}{15}\cos^{-1}\left[\frac{\sin(-0.833°) - \sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] \quad (26)$$

where Z denotes Zuhr prayer time (Eq. 14) and the rest have their defined meanings.

Isha'a & Fajr Prayer Times: As explained in section 2.1, the Isha'a and Fajr prayers begin at the start of the evening and true morning twilights, respectively. These twilights are mirror images of one another and correspond to certain Sun depression angles after and before the sunset or sunrise. Nonetheless, there is no precise and universal depression angle for Isha'a/Fajr times across the board and different views exist for the best and the practical angle depending on the observer's region and/or country. Table 1.0 shows the different depression angles adopted by various institutions [19]. The higher the angle, the earlier the Fajr and later the Isha'a times.

In our calculations, we adopted a depression angle of α = -18° and the equations for Isha'a and Fajr prayers are shown below (Note: *-ve for Fajr and +ve for Isha'a*):

$$Fajr\ (F) = Z - \frac{1}{15}\cos^{-1}\left[\frac{\sin(-18°) - \sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] \quad (27)$$

$$Isha'a\ (I) = Z - \frac{1}{15}\cos^{-1}\left[\frac{\sin(-18°) - \sin(\delta)\sin(\varphi)}{\cos(\delta)\cos(\varphi)}\right] \quad (28)$$

where Z, δ and φ have their previously defined meanings.

RESULTS AND DISCUSSION

For the "*proof-of-concept*" purpose, this study focuses on the understudied area of East Africa region i.e. Somalia, Djibouti, Ethiopia and Kenya benchmarked

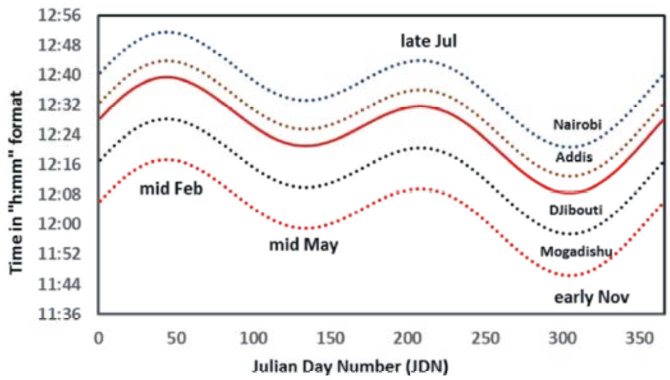

Fig. 4: Zuhr start times for all locations studied. The red solid curve represents Mecca and the rest denote the cities as labelled

Table 2.0: Contains all the geographical coordinates for the capitals of East Africa region plus Mecca for reference. Note all of the altitudes are on the northern hemisphere apart from Nairobi which is just below the equator in the southern hemisphere

Country/City	Longitude (λ)	Latitude (φ)
Saudi Arabia / Mecca	39.8591° E	21.3891° N
Somalia / Mogadishu	45.3182° E	2.0469° N
Djibouti / Djibouti	42.5903° E	11.8251° N
Ethiopia / Addis Ababa	38.7578° E	8.9806° N
Kenya / Nairobi	36.8219° E	1.2921° S

against the prayer times for Mecca. The results presented are – hereby – computed for the capital cities of the above-mentioned countries with the geographical coordinates shown in Table 2.0 – note all the cities studied including Mecca have the same time zone of "*UTC + 3*" and their latitudes are on the north of the equator except for Nairobi which is on the southern hemisphere just below the equator.

Zuhr Prayer Times: Fig. 4 shows the Zuhr prayer times throughout the year for all the locations studies based on their various geographical coordinates using Eq. 14. Two factors have direct effect on the timing of Zuhr prayer: the local meridian (longitude) angle and the effect of the Equation of Time (eccentricity and obliquity effects) responsible for the sinusoidal time variation of Zuhr prayer with the days and months of the year (Fig. 3 & Eq. 13).

Geographically, all of these locations occur east to the Prime Meridian (i.e. the universal standard meridian with longitude value of $\lambda = 0°$) and the angles for their corresponding longitude increases in the following order (Table 2.0): $\lambda_{Mogadishu} > \lambda_{Djibouti} > \lambda_{Mecca} > \lambda_{Addis} > \lambda_{Nairobi}$. Accordingly, the start of Zuhr time for these cities shifts up during the day in the opposite order: Nairobi >Addis> Mecca > Djibouti > Mogadishu (i.e. from cities with lowest λ value to highest; Fig. 4). For all cities, the start of Zuhr time is highest in mid-February and lowest in early November due to the effect of orbit eccentricity and ecliptic obliquity of the earth's motion around the Sun. Moreover, Zuhr prayer is independent of latitude and Sun declination angles and changes solely with local meridian of particular locations. As noted earlier, the Zuhr time ends – in the view of the majority of Islamic Jurisprudence schools – when the length of the shadow reaches the same height as the objects. This also marks the beginning of Asr prayer.

Asr Prayer Time: Tracing the Shadow: The key aspect in determining the beginning of Asr prayer time is to measure the relative shadow length of objects. Since all the cities under study fall within the two tropics (the tropic of Cancer and the tropic of Capricorn), there are two distinctive days for each location where there is no zenith shadow at noon time i.e. when the Sun is directly overhead.

Fig. 5 shows the normalized relative shadow length at zenith time defining the ratio of "zenith shadow-to-object height" throughout the year at noon time; the ratio is 1 when the shadow is equal in height to the object and 0 when there is no zenith shadow. From the data illustrated in Fig. 5, the length of the shadow of objects as a function of Julian day number (JDN) can be traced and its value can be retrieved for each day of the year with ease. Mecca (the red solid line) falls on the highest latitude in northern hemisphere and has the highest relative shadow length of all locations during winter time when the Sun is in the southern hemisphere where Nairobi – on the southern hemisphere - has the lowest relative shadow length in the same period. The trend of relative shadow lengths for all

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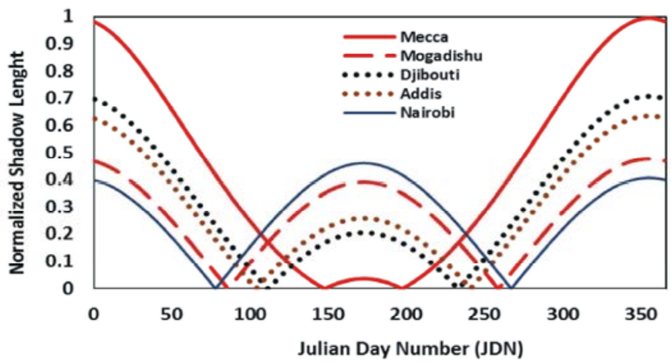

Fig. 5: Normalized Relative Length of objects' shadow at the different locations studied as a function of Julian Day Number. The red solid line represents Mecca, the dashed red line represents Mogadishu, the dotted black line represents Djibouti, the brown dotted line represents Addis Ababa and the solid blue line represents Nairobi

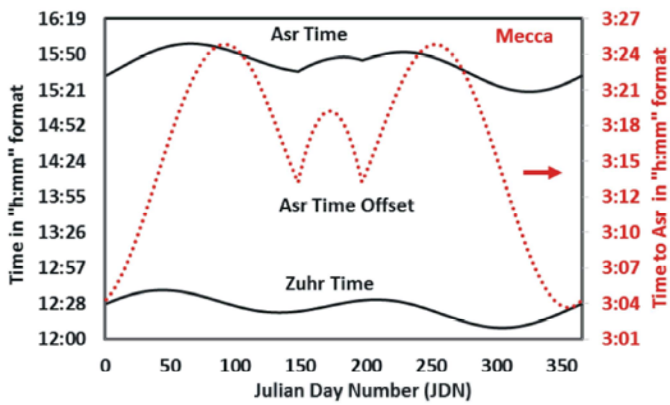

Fig. 6: Shows Zuhr prayer time, time to Asr (Shadow Offset time) and the combination of the two marking both the beginning and end of Zuhr and Asr times when the shadow reaches the same height as that of objects. The red dotted curve denotes the Asr offset time and its corresponding time span is marked on the right hand y-axis

locations is consistent with the values of latitudes (φ) shown in Table 2.0 increasing in the order: φ = 21.4°> 11.8°> 9°> 2.0°> 1.3° (correct to 1 d.p) from Mecca (highest latitude) to Nairobi (lowest latitude).

Interestingly, in December 21st i.e. winter solstice, the zenith shadow has the same length as that of objects for Mecca (ca. 100%) and Djibouti has the second highest shadow length (ca. 70%). Conversely, the zenith shadow reaches its highest length for Nairobi (up to 50% of object's height) during the summer time when the Sun is in the northern hemisphere. At this time, in the summer solstice, June 21st, Mecca has the lowest zenith shadow (ca. 4-5%).

One can easily discern from the data represented in Fig. 5 that there is no zenith shadow (ca.0% relative height) in the transition from spring to summer coinciding with the equinox days on 19th March, 27th March, 14th April, 22nd April and 27th May for Nairobi, Mogadishu, Addis Ababa, Djibouti and Mecca, respectively as well as in the transition from autumn to summer equinoxes on 16th July, 21st August, 29th August, 16th September and 24th September for the same locations in reverse order (from Mecca to Nairobi). This happens when the Sun is directly overhead in respective cities.

Further to this, the beginning of Asr prayer time is determined from the sum of the Zuhr time - depending on the local meridian degree - and the time it takes the shadow to reach the same height as that of objects - apart from the zenith shadow - governed by the Sun declination angle and latitude angle of particular locations.

This combination of factors makes the Asr time fluctuate irregularly and non-monotonically with Julian days of the year going through different maxima and minima. Fig. 6 portrays the Mecca Asr time, Zuhr time, the time it takes for the shadow to reach the same height as that of objects (as a result of the hour angle as described in Eq.23) as well as the resultant curve from the combination of the two components (Zuhr and Asr Offset times) marking both the end of Zuhr and the beginning of Asr prayer times (Eq. 24).

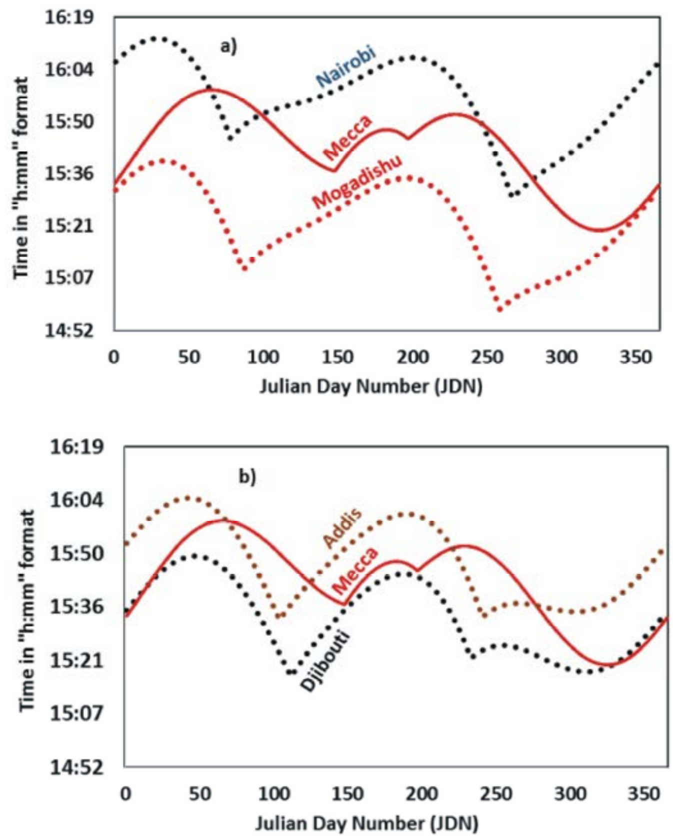

Fig. 7: Shows the beginning of Asr prayer times – by default the end of Zuhr times for the four cities: Mogadishu and Nairobi in panel (a) and Djibouti and Addis Ababa in panel (b). Both panels show Mecca Asr timing as a reference

For the special case of Mecca - as in Fig. 6 - the Asr offset time peaks in the spring and autumn times, slightly decreases in the summer time and significantly falls down to its minimum point in the winter (December month) when the day time is shortest. The two distinctive and intermediate minima points correspond to the Julian days of 148 (27th May) and 197 (16th July) when the Sun is directly overhead in Mecca.

The beginning of Asr prayer times for the rest of cities are shown in Fig. 7a & b. In this figure, we paired the cities with similar curve shape and analogous patterns influenced by their latitudes falling within the same vicinity; Mogadishu and Nairobi show similar behaviour (φ = 2.0° and φ = 1.3°, respectively) and similarly, Djibouti and Addis Ababa have analogous pattern (φ = 11.83° and φ = 8.98°, respectively).

Maghrib and Isha'a Prayer Times: For Maghrib and Isha'a prayers, the effect of local latitude (φ) is more dominant in determining the overall shape and the fringe pattern (rising and falling) of the curve while the effect of the local meridian (λ) controls the height or depth (amplitude) of curve fringes i.e. the time span of the day - based on the time the Sun sets or rises for each day of the year.

Considering the impact of the interplay between these two parameters (λ and φ) on the overall behaviour of the prayer time curve, we paired – as we did for Asr - Mogadishu (φ = 2.0° N) with Nairobi (φ = 1.3° S) in panels "a" of Fig. 8 and Fig. 9 for Maghrib and Isha'a, respectively. The prayer times for Maghrib and Isha'a for Addis Ababa (φ ~ 9.0° N) and Djibouti (φ ~ 11.9° N) are also shown in panels "b" of Fig. 8 and Fig. 9, correspondingly.

Fajr Prayer Times: Quite interestingly but not surprisingly, the shape and the fringe pattern for Fajr prayer times as a function of Julian day number is the 180° reflection of a perfect mirror image of Isha'a times as represented in Fig. 10. This is due to the fact that the same depression angles of the Sun below the horizon in terms

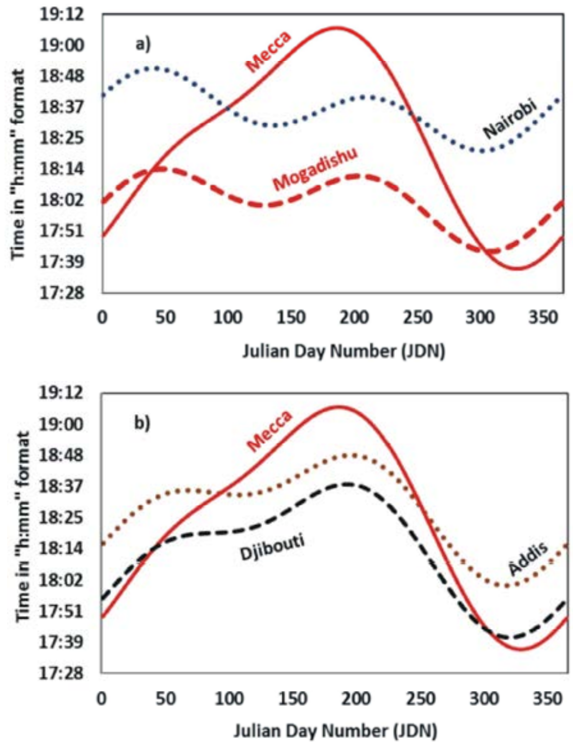

Fig. 8: Shows the beginning of Maghrib prayer times. Mogadishu and Nairobi prayer times in panel (a) and Djibouti and Addis Ababa in panel (b). Both panels show Mecca Maghrib timing as a reference

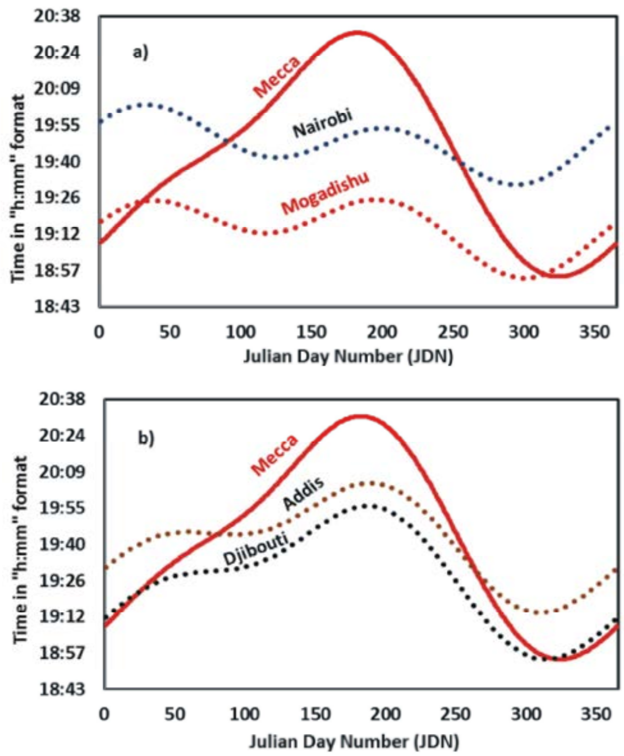

Fig. 9: The beginning for Isha'a prayer times. Panel "a" shows the Isha'a times for Mogadishu and Nairobi and panel "b" shows the Isha'a prayer times for Addis Ababa and Djibouti

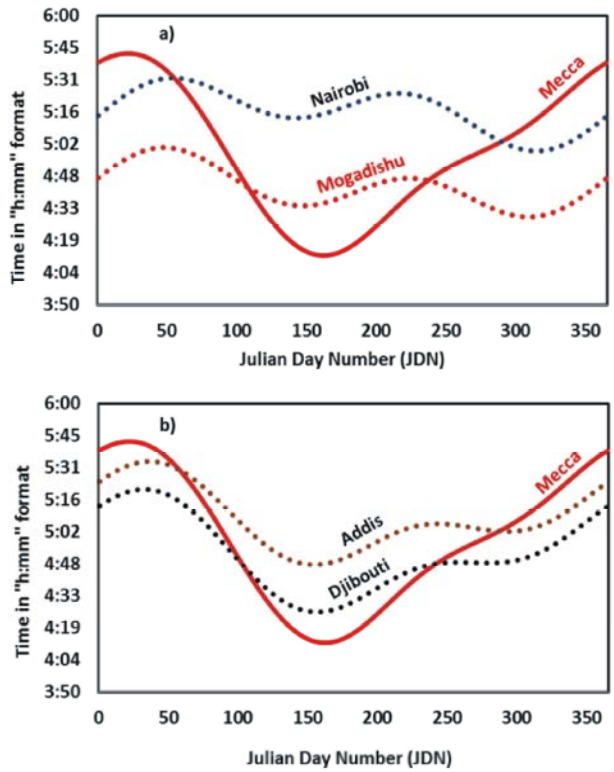

Fig. 10: Fajr Prayer times: panel "a" shows Mogadishu and Nairobi Fajr prayer times and panel "b" shows the prayer times for Addis Ababa and Djibouti. The red solid line denotes the Mecca Fajr prayer times as clearly labelled in the panels.

of magnitude (|α| = 18°) but with opposite signs were used in calculating Isha'a and Fajr prayers. The diffuse and bright light scattering as a result of the evening and morning twilights are caused by the Sun either moving away or towards the horizon at the same depression angle (α = -18), correspondingly (refer to Eqs. 27 & 28).

Similar to Isha,a and Maghrib times, panel "a" of Fig. 10 shows the Fajr times for Mogadishu and Nairobi in addition to Mecca Isha'a times as a reference. The Fajr times for Djibouti and Addis Ababa are also shown in panel "b" of Fig. 10.

CONCLUSIONS

The mathematical determination of Islamic prayer times is key for high latitude locations and extremely helpful for locations with low-to-medium range latitudes where the Sun movement can easily be observed with naked eye. In this work we have used spherical trigonometry functions in calculating all prayer times for the main cities of East African counties. In addition, the prayer times for Mecca were computed to serve as benchmark and comparison purpose. For data analysis and computations, the widely available but powerful MS Excel programme was used. It has been ostensibly shown that prayer times varies in non-monotonic and periodic pattern as a function of Julian day number (JDN) of the year. The factors causing the shape, fringes and the amplitudes of the cyclic curves for prayer times have been discussed in depth. The latitude coordinate – for example - of locations has significant effect in the cyclic fluctuation (rise and fall positions) of prayer timing during the year but the height and the amplitude of the fringes depends on the longitude variable controlling the length of the day (the period between sunrise and sunset).

For Asr prayer, in particular, determining the relative shadow length of objects at zenith time was modelled for the first time – to the best of our knowledge. There is always a zenith shadow at noon time for all locations except the equinox days (vernal and autumnal) in-between the tropics. Knowing the length of zenith shadow is a prerequisite for the calculation of the beginning of Asr prayer as well as for the end of Zuhr prayer. Similarly, all mathematical equations required for computing all prayer times were thoroughly reviewed in great detail. The hour angles for the time offset of all other prayers with reference to Zuhr prayer were calculated depending on the Sun declination angle, elevation angle and corresponding geographical coordinates of particular locations.

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