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Existence of Solutions for Non-Smooth Quasi-Convex Functions in a Real Reflexive Banach Space

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Abstract: It is known in literature that if the objective functions f of a problem is convex and smooth, existenceof global solution are guaranteed but in real world, there are some problems in which their objective functions are non-convex and are more general than convex functions. In this paper, Existence of solutions for a generalized convex function called quasi-convex function, in particular non-smooth quasi-convex function in a real reflexive Banach space was discussed. This was established by restricting the objective function to be lower semi-continuous (lsc). The paper also discussed a case where the effective domain $D(f)$ is not necessarily bounded, but if f is proper and sequentially coercive, existence of solution is still guaranteed. This work equally showed that if f is quasi-convex and lower semi-continuous then, f is weakly lower semi continuous.

Key words: Lower semi-continuous · Quasi-convex function Reflexive Banach space · Sublevel set · Weakly closed

INTRODUCTION

Generally, in optimization problems, the question on existence is very important. This is because not all optimization problems have solutions. The question is, under what condition(s) on the objective function f or the constraint set D are we guaranteed that solution will always exists for the optimization problems?.

It is known that if the effective domain $D(f)$ is finite dimensional space, D– compact and f is continuous, then optimization problem has at least one solution. However, in real life problems such as optimal design problem, optimal control problem, X need not be in a finite dimensional space, the objective function f need not be continuous and D may not necessarily be compact

Authors such as [1], [2], Juan [3] discussed intensively in finite dimensional spaces. Authors such as, Alen [4], Chidume [5] and Burger [6] worked on Hilbert Space, Reflexive space and optimal Design respectively, restricting themselves to convex functions. The objective of this paper is to provide conditions that will guarantee existence of solution for non-smooth quasi-convex functions in a real reflexive Banach space.

Definition 1.1: Let X be a real Banach space and D be subset of X. The set D is said to be convex if for each x,y in D and for each $\lambda \in [0,1]$, we have

$$\lambda x + (1 - \lambda) y \in D$$

Definition 1.2:

- A function, $f{:}D \subset X \to \mathbb{R} \cup \{+\infty\}$ is convex if for each $\lambda \in [0,1]$ and for each $x, y \in D$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1- \lambda)f(y)$$

- Geometrically, a function $f{:}D \subset X \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if and only if the $epi(f) = \{(x, \alpha) \in X \times \mathbb{R}: f(x) \leq \alpha\}$, is a convex set.

- f is called proper if there exists at least one $x_0 \in D$ such that $f(x_0) \neq \infty$

Minimization of Quasi-Convex Function: Before we present the main results, we recall some definitions and important results on quasi-convex functions.

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Definition 2.1: Let f: $D \subset X \to \mathbb{R} \cup \{+\infty\}$, then f is said to be quasi-convex if and only if for each $\lambda \in [0,1]$ and for each $x, y \in D$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}.$$

Definition 2.2:

- A function f: $D \subset X \to \mathbb{R} \cup \{+\infty\}$, is lower semi-continuous at $x \in X$ if for $\{x\} \in X$ converges to $x \in X$, we have,

$$f(x) \leq \lim_{n\to\infty} \inf f(x_n)$$

- Geometrically, a function, f: $D \subset X \to \mathbb{R} \cup \{+\infty\}$, is said to be lower semi-continuous if and only if the sublevel set,

Sublevel set

$$\overline{l}_\alpha(f) = \{x \in X : f(x) \leq \alpha, \alpha \in \mathbb{R}\},$$

is a closed set.

Definition 2.3:

- A function f: $D \subset X \to \mathbb{R} \cup \{+\infty\}$, is weakly lower semi continuous (wlsc) if,

$$f(x) \leq \lim_{n\to\infty} \inf f(x_n)$$

for all sequences $\{x\} \rangle X$ converges weakly to $x \in X$.

- Geometrically, f: $D \subset X \to \mathbb{R} \cup \{+\infty\}$, is weakly lower semi continuous if the sublevel set is weakly closed.

Lemma 2.1: Let X be a real Reflexive Banach space and f: $D \subset X \to \mathbb{R} \cup \{+\infty\}$ then f is said to be quasi-convex if and only if the sublevel set,

$$l_\alpha(f) = \{x \in X: f(x) \leq \alpha, \alpha \in \mathbb{R}\}.$$

is a convex set.

Proof:
First suppose that f is quasi-convex function, then we show that $l_\alpha(f)$ is a convex set for each

$\alpha \in \mathbb{R}$ let $x \in l_\alpha(f)$, $y \in l_\alpha(f) \Rightarrow f(x) \leq \alpha, f(y) \leq \alpha$,

Since f is quasi-convex, for $\lambda \in (0,1)$ and for each x, y $\in$ D, we have

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}. \qquad (2.1)$$

Assume without loss of generality that

$$f(x) \leq f(y), \qquad (2.2)$$

Since,

$$f(y) \leq \alpha, \qquad (2.3)$$

we have

$$f(\lambda x + (1 - \lambda)y) \leq \alpha. \qquad (2.4)$$

This implies,

$$\lambda x + (1 - \lambda)y \leq \alpha \in l_\alpha(f) \qquad (2.5)$$

Hence $l_\alpha(f)$ is convex.

Conversely, suppose $l_\alpha(f)$ is convex, we show that f is quasi-convex let $x, y \in X$, $\lambda \in [0,1]$ then $x, y \in X$, implies that $f(x) \in \mathbb{R}$, $f(y) \in \mathbb{R}$. Thus, $(x, f(x) \in l_\alpha(f))$ and $(y, f(y) \in l_\alpha(f))$

But, $l_\alpha(f)$ is convex, thus,

$$\lambda(x, f(x) + (1-\lambda)(y, f(y)) \in l_\alpha(f)$$
$$\Rightarrow (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in l_\alpha(f)$$
$$\Rightarrow f(\lambda x + (1-\lambda)y) \leq \lambda \max\{f(x), f(y)\} + (1-\lambda)\{f(x), f(y)\} \qquad (2.6)$$

$$\Rightarrow f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}. \qquad (2.7)$$

Hence, f is quasi-convex

Lemma 2.2: (Eberlin smul'yan theorem): A real Banach space X is reflexive if and only if every bounded sequence in X has a subsequence which converges weakly to an element of X.

See Chidume (2014) for the proof.

Definition 2.4: Let X, be a real reflexive space. A function f: $X \to \mathbb{R} \cup \{+\infty\}$ is said to be coercive if.

$\lim_{\|x\|\to+\infty} f(x) = +\infty$.

Lemma 2.3: (Hahn Banach) Let $F \subset X$ and $G \subset X$ be two convex sets, nonempty and disjoint. Suppose that F is open. Then, there exists a closed hyperplane that separates F and G in a general sense. See Chidume (2014), Brezis (1983) for the proof.

Lemma 2.4: Let $D \subset X$ be closed (strongly) and convex set then, D is weakly closed.

Proof: To prove this, we need to show that D^c is weakly open. Let $x_0 \in D^c$ then $x_0 \notin D$. From the hypothesis, $\{x_0\}$ is convex, nonempty and compact and D is nonempty convex and closed, by lemma 2.3, there exist $f \in X^*$, $t \in \mathbb{R}$ such that $f(x_0) = \langle f, x_0\rangle < t < \langle f, x\rangle$, $\forall\, x \in$ D.

If we construct a set of the form

$V = \{x \in X: \langle f, x\rangle < t$

Then, $x_0 \in V$, $V \cap$ D $= \varnothing$, so that $V \subset D^c$ more so V is weakly open.
Hence D^c is weakly open and so D weakly closed.

RESULTS

Consider the problem,

Min $f(x)$

Subject to $x \in D \subset X$
where, X is a real reflexive Banach space, $f: X \to \mathbb{R} \cup \{+\infty\}$ is quasi-convex and non-smooth function and D is a convex set,.

Before, stating the main results, we state the following proposition that will be of help in the proof of the results

Proposition 3.1: Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be any map. Then, if f is quasi-convex and lower-semi continuous, then f is quasi-convex and weakly lower semi-continuous.

Proof: Suppose that f is quasi-convex function and lower semi continuous, we prove that f is quasi-convex and weakly lower semi-continuous. To show that, it suffices to show that the sublevel set, $l_\alpha(f)$, is convex and closed. f is quasi-convex implies by lemma 2.1 that l_α (f) is a convex set. We now show that l_α (f) is closed.

Let, $(x_n, \alpha_n) \in l_\alpha$ (f) such that

$(x_n\ \alpha_n) \to (x, \alpha)$,

By the definition of sub level set

$$f(x_n) \leq \alpha_n. \tag{3.1}$$

S ince f is lower semi continuous

$$\Rightarrow f(x) \leq \lim_{n\to\infty} inf f(x_n) \leq \lim_{n\to\infty} inf(\alpha_n) = \lim_{n\to\infty} \alpha_{n=\alpha}. \tag{3.2}$$

$$\Rightarrow f(x) \leq \alpha. \tag{3.3}$$

So, $(x, \alpha) \in l_\alpha$ (f),

Hence, $l_\alpha(f)$, is closed
Since, $l_\alpha(f)$ is convex and closed by lemma 2.4, $l_\alpha(f)$ is weakly closed and so f is weakly lower semi-continuous and quasi-convex.

Theorem 3.1: Let D be a closed convex bounded and nonempty subset of X and X is a real reflexive Banach space. Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous and quasi- convex function. Then f has a minimizer at $x^* \in$ D
That is,

$$f(x^*) = inf_{x \in D} f(x) \tag{3.4}$$

Proof:
We note that D being close and bounded here does not mean that D is compact.
We now use the following steps for the proof
Step1. Let $b = \inf_{x \in D} f(x)$
Let us suppose that, $b = -\infty$.
Then $\exists\, x_m \in$ iD such that

$$f(x_n) < -m, \forall\, m \in \mathbb{N} \tag{3.5}$$

Step II: Since D is bounded, (x_m) is bounded because, $x_m \in D$ and since X is a reflexive Banach space, by lemma 2.2, we can have a subsequence, (x_{mk}) of (x_m), that converges weakly to some point x^* on X.

Step III: Since D is convex and closed, by lemma 2.4, D is weakly closed and we have, $x^* \in D$. Since f is quasi-convex and lower semi-continuous, by proposition 3.1, f is weakly lower semi-continuous at x^*, that is,

$$f(x) \leq \lim_{k\to\infty} inf f(x_{mk}) \quad (3.6)$$

by (3.5), we have,

$$f(x) \leq \lim_{k\to\infty} inf f(x_{mk}) < -\infty \quad (3.7)$$

Which is a contradiction, since, $f(x) \in \mathbb{R} \cup \{+\infty\}$ and so $b \in \mathbb{R}$

Step IV: Using the definition of infimum, there exist, $\{x_m\} \to x^*$, $m \in \mathbb{N}$ and

$x^* \in D$ such that

$$b \leq f(x_m) < b + \delta, \text{ Let } \delta = \frac{1}{m} \quad (3.8)$$

we have

$$b \leq f(x_m) < b + \frac{1}{m} \text{ as } m \to \infty$$

$$b \leq f(x^*) \quad (3.9)$$

$(x_m) \in D$, implies $\{x_m\}$ is bounded and then, there exists a subsequence $\{x_{mk}\}$ of $\{x_m\}$ and $x^* \in D$ such that $\{x_{mk}\}_{k\in\mathbb{N}}$ converges weakly to x^*.

Step V: Using the fact that, f is lower-semi-continuous and quasi-convex, we have that f is weakly lower semi-continuous and we obtain,

$$f(x^*) \leq \lim inf_{k\to\infty} f(x_{m_k}) \leq \lim_{k\to\infty} inf \left(b + \frac{1}{m_k} \right) =$$

$$\lim_{k\to\infty} (b + \frac{1}{m_k}) = b \quad (3.10)$$

$$\Rightarrow f(x^*) \leq b$$

Combining (3.9) and (3.10)

We obtain,

$$f(x^*) = b = inf_{x\in D} f(x) \quad (3.11)$$

Hence, x^* is a minimizer of f on D.

Before we present the second result on existence we recall the following definitions

Definition 4.1: A function, f: $X \to \mathbb{R} \cup \{+\infty\}$, is said to be weakly sequentially coercive if
$\lim_{\|x\|\to\infty} f(x) = +\infty$

Definition 4.2: A function, f, is said to be proper if there exists at least one $x_0 \in D$ such that if,$(x_0) \neq +\infty$ i.e $f(x_0) \in \mathbb{R}$.

Now, we present second result of this section Suppose D lossed boundedness and f is proper lower semi-continuous function and coercive. We prove that, there exists a solution to problem (Q) under

Theorem 4.2: Let f: $X \to \mathbb{R} \cup \{+\infty\}$ be quasi convex proper lower semi-continuous function and let X be a real reflexive Banach space. Suppose if is sequentially coercive. Then, f has a minimizer at $x^* \in D$. That is, there exists $x^* \in X$ such that

$$f(x^*) \leq f(x), \forall x \in X,$$
$$\Rightarrow f(x^*) = inf_{x\in X} f(x)$$

Proof: We prove using the following steps:

Step 1: f is proper, implies that, there exists $x_0 \in X$ such that,

$$f(x_0) \neq +\infty.$$

Now, we construct a set D and show that it is nonempty closed convex and bounded subset of X and apply theorem 3.1

Consider, f: $D \subset X \to \mathbb{R} \cup \{+\infty\}$ with

$$D. = \{x \in X: f(x) \leq f(x_0)\} \quad (3.12)$$

To show that, we know that D is a section with $t = f(x_0)$, this implies that f is quasi convex and lower semi-continuous, by lemma 2.1 and definition 2.2(ii), D is convex and closed.

Step 2: We now assume that D is bounded. To prove our assumption, we show by contradiction. Suppose D is not bounded, then there exist a sequence $x_m \in D$ such that

$$\| x_m\| < m, \forall m \in \mathbb{N}.$$

Since, $x_m \in$ iD,

we obtain,

$$f(x_m) \leq f(x_0), \|x_m\| > m \quad (3.13)$$

$$\lim_{\|x\|\to\infty} f(x) = +\infty, \quad (3.14)$$

$\lim_{m\to\infty} \| x_m \| = +\infty$,

by the hypothesis

$$\lim_{m\to\infty} \|x_m\| = +\infty \quad (3.15)$$

Contradicting inequality, (4.9)

Hence D is bounded. By theorem 4.1, $\exists\, x^* \in D \subset X$ such that

$f(x^*) \leq f(x), \forall\, x \in D$.

Suppose $x \in X \backslash D$, then, $f(x) > f(x_0)$
Since, $x_0 \in D$, $f(x^*) \leq f(x_0)$
Thus, $f(x^*) < f(x)\ \forall x \in X$

$$\Rightarrow f(x^*) \leq f(x), \forall x \in X \quad (3.16)$$

Hence,

$f(x^*) = inf_{x\in X} f(x)$

CONCLUSION

Herein, we have discussed problem in which their objective functions are non-convex and non-smooth which are more general than convex functions. In particular, this paper, discussed Existence of solutions for a generalized convex function called quasi-convex function, in a real reflexive Banach space. This was established by restricting the objective function to be lower semi-continuous (lsc) and using some results in weak topology. The paper also discussed a case where the effective domain $D(f)$ is not necessarily bounded, but if f is proper and sequentially coercive, existence of solution is still guaranteed. This work equally revealed that if f is quasi-convex and lower semi-continuous then, f is weakly lower semi continuous.

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