Middle-East Journal of Scientific Research 19 (12): 1722-1728, 2014
ISSN 1990-9233

DOI: 10.5829/idosi.mejsr.2014.19.12.21846

Some Properties of ß-Continuous Mappings, ß-Open Mappings and ß –Homeomorphism

Yasser A. Abou-Elwan

Department of Mathematics, Faculty of Science, University of Omer Almukhtar

Abstract: In this paper, we construct and study several new properties and characterizations for B-continuous mappings, B-open mappings and B-homeomorphism. The connections between these sorts of non-continuous mappings and various kinds of mappings as almost – continuous sense of Signal and Signal (a. o . s.) , weakly continuous, thickening, almost – continuous sense of Signal and Signal (a. o . s.) are investigated. The concept of a ß- topological property, for example the property of being a topological space be of the first category is one of ß- topological properties, i.e. it is invariant under a ß-homeomorphism.

Key words: B-continuous mappings . B-open mappings . B-homeomorphism . a. o. s. . ß-compact . thickening . bi-M-ß-continuous . ß-topological property

INTRODUCTION

A subset A of the space X is called a semi-open (resp. a-set , pre-open , ß-open , regular open) $A \subset A^{o-}$ (resp. $A \subset A^{o-o}$, $A \subset A^{-o}$, $A \subset A^{-o-}$, $A = A^{-o}$). The complement of a semi-open set (resp. a-set, pre-open, ß-set, regular open set), is called a semi-closed [1] (resp. a- closed , pre- closed , ß- closed, regular closed). The family of all semi-open (resp. a-set, pre-open, ß-open, regular open) sets of a space X will be denoted by SO(X) (resp. a(X), PO(X), ßO(X), RO(X)).

A mapping f : X $\rightarrow$ Y is called semi-continuous (resp. a-continuous , pre-continuous and ß- continuous) if the inverse image of every open set in Y is semi-open (resp. a-set, pre-open, ß- open) in X.

Theorem 1.1. Let f : X $\rightarrow$ Y be a mapping, then the following statements are equivalent:

i) f is ß-continuous.

ii) For every x $\in$ X and every open set V $\subset$ Y containing f(x), there exists a ß-open set W $\subset$ X containing x such that f(W) $\subset$ V[2].

iii) The inverse image of each closed set in Y is ß-closed in X.

iv) $(f^{-1}(B))^{o-o} \subset f^{-1}(\overline{B})$, for every B $\subset$ Y.

v) $f(A^{o-o}) \subset \overline{(f(A))}$, for every A $\subset$ X.

Theorem 1.2. Let f: X $\rightarrow$ Y be a mapping, then the following statements are equivalent:

i) f is ß-open.

ii) For every x $\in$ X and every neighborhood U of x, there exists a ß-open set W $\subset$ Y containing f(x) such that W $\subset$ f(U).

iii) $f^{-1}(B^{o-o}) \subset \overline{(f^{-1}(B))}$, for every B $\subset$ Y.

iv) If f is bijective, then $(f(A))^{o-o} \subset f(\overline{A})$, for every A $\subset$ X.

A mapping f: X $\rightarrow$ Y is said to be M-ß-continuous , if the inverse image of each ß-open set in Y is ß-open in X. Also, f is said to be M-ß-open , if the image of every ß-open set in X is ß-open in Y.

Two spaces X and Y are called ß-homeomorphic if there exists a bijective mapping. f : X $\rightarrow$ Y such that f is M-ß-continuous and M-ß-open. Such a mapping is called a ß-homeomorphism[3].

Corresponding Author: Yasser A. Abou-Elwan, Department of Mathematics, Faculty of Science, University of Omer Almukhtar, Libya.

B-CONTINUOUS MAPPINGS

Theorem 2.1. A mapping f : X → Y is ß-continuous, iff $f(\overline{U}^{\circ}) \subset (f(U))^{-}$, for every open set $U \subset X$.

Proof. Let f be a ß-continuous , then by Theorem 1.1.4 [10] , $f(U^{\circ-\circ}) \subset (f(U))^{-}$. Since $U \subset X$ is open, $f(U^{-\circ}) \subset (f(U))^{-}$.

Conversely, let $V \subset Y$ be an open set , W = Y – V and A = $(f^{-1}(W))^{\circ}$. Then , $f((f^{-1}(W))^{\circ-\circ} \subset f(((f^{-1}(W))^{\circ})^{-} \subset (f(f^{-1}(W)))^{-} \subset \overline{W} = W$. so $(f^{-1}(W))^{\circ-\circ} \subset f^{-1}(W)$. Hence $f^{-1}(W)$ is ß-closed and f is ß-continuous[4] .

Definition 2.1. A space X is called ß-compact if every ß-open cover of X has a finite subcover.

Definition 2. 2. A space X which has the property Any open cover of X has a finite subfamily, the closure of whose members cover the space. Equivalently, any open cover of X has a finite proximate subcover, any open cover of X has a finite AU – property and any open cover of X has a finite subfamily whose union is dense in Xis called H(i) space by C.T. Scarbourough and
A.H. Stone, almost compact space by M.k. Singal and Asha Rani, generalized absolutely closed by Chen Tung Liu and quasi H-closed (Q H C) by J. Porter and J. Thomas .

Now , we introduce the following theorem .

Theorem 2.2. If f : X → Y is a ß-continuous surjection from a ß-compact space X onto a space Y and $(f^{-1}(V))^{-\circ-} \subset f^{-1}(\overline{V})$, for every open set $V \subset Y$, then Y is almost compact .

Proof. Let ζ be an open covering of Y. Then $\{f^{-1}(V) : V \in \zeta\}$ is a ß-open cover of X . Since X is ß-compact , so there exists a finite subfamily $\{V_1, \dots, V_n\}$ of ζ such that X $= \bigcup_{i=1}^{n} f^{-1}(V_i) \subset \bigcup_{i=1}^{n} (f^{-1}(V_i))^{-\circ-} \subset \bigcup_{i=1}^{n} \left(f^{-1}(\overline{V}_i)\right) = f^{-1}(\bigcup_{i=1}^{n} \overline{V}_i)$. Since f is surjective , $f(X) = Y = \bigcup_{i=1}^{n} \overline{V}_i$. Therefore , Y is almost compact[5] .

Remark 2.1. ß-continuity does not preserve , in general , connectedness , as is illustrated by the following example.

Example 2.1. Let X = Y = { 0 , 1} and the topology on X is indiscrete and on Y is discrete . Then the identity mapping i: X → Y is ß-continuous surjection and X is connected , but i(X) = Y is not connected .

Now we give the following theorem:

Theorem 2. 3. If f: X → Y be a ß-continuous surjection from a connected space X onto a space Y and $(f^{-1}(V))^{-\circ-} \subset f^{-1}(\overline{V})$ for every open set $V \subset Y$, then Y is connected,

Proof. Assume that Y is not connected, then there exist two open subsets V_1 and V_2 of Y, such that $V_1 \cup V_2 = Y$ and $V_1 \cap V_2 = \emptyset$. Thus V_1 and V_2 are both open and closed . By hypothesis , $(f^{-1}(V_i))^{-\circ-} \subset \left(f^{-1}(\overline{V}_i)\right)$ i = {1,2} . Since f is ß–continuous , $f^{-1}(V_i) \subset (f^{-1}(V_i))^{-\circ-} \subset f^{-1}(\overline{V}_i) = f^{-1}(V_i)$, so , $(f^{-1}(V_i))^{-\circ-} = f^{-1}(V_i)$ for each i = {1, 2} .
Hence $(f^{-1}(V_1))^{-\circ-} \cup (f^{-1}(V_2))^{-\circ-} = X$ and $(f^{-1}(V_1))^{-\circ-} \cap (f^{-1}(V_2))^{-\circ-} = \emptyset$
and this shows that X is not connected . Therefore Y is connected if X is connected .

Remark 2. 2. Also, Example 2.1. illustrates that the condition $(f^{-1}(V))^{-\circ-} \subset f^{-1}(\overline{V})$ for every open set $V \subset Y$ is the above theorem is necessary [6].

Definition 2. 3. A mapping f : X $\to$ Y is almost continuous in the sense (briefly,a.c.s.) of Singal and Signal (resp. θ-continuous , weakly continuous) if for each x $\in$ X and each open set open set V $\subset$ Y containing f(x) , there exists an open set U $\subset$ X containing x such that f(U) $\subset \overline{V}^{\circ}$ (resp. f($\overline{U}$) $\subset \overline{V}$, f(U) $\subset \overline{V}$) .

The following diagram gives the connections between these types of mapping .

Continuity $\to$ (a.c.s.) $\to$ θ-continuity $\to$ weak continuity.

The converse of these implications are not true , in general .

The following examples illustrates that almost-continuity and ß-continuity are independent.

Example 2. 2. Let f: X $\to$ Y be an injection from an excluding point topological space X with excluding point p into a particular point topological space with particular point f(p) , then f is a.c.s, but not ß-continuous .

Example 2.3. Let f: X $\to$ Y be an injection from a particular point topological space X with particular point p into excluding point topological space Y with excluding point f(p), then f is ß-continuous and is not a.c.s [7-10].

Now, since a.c.s $\to$ θ-continuity $\to$ weak continuity then the above examples show that θ-continuity or weak continuity may not be ß-continuity. See also, the following example.

Example 2.4. Let f : R $\to$ R be a mapping on the real line R given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational ,} \\ -x & \text{if } x \text{ is irrational .} \end{cases}$$

It is clear that f is ß-continuous , but not weak continuous and , again , this shows that ß-continuity may not be θ-continuity or a.c.s.

Theorem 2. 4. . A pre-open mapping which is weakly continuous is pre-continuous .

Proof. See.

Now, the following theorem gives the connection between ß-continuity and weak continuity.

Theorem 2.5. A pre-open weakly continuous mapping is ß-continuous.

proof. It is obvious.

Corollary 2.1. A pre-open, a.c.s. mapping is ß-continuous[11-14].

B-OPEN MAPPINGS.

Theorem 3.1. An injective mapping f: X $\to$ Y is ß-open if and only if $f^{-1}(\overline{B}^{o}) \subset (f^{-1}(B))^{-}$, for each open set B $\subset$ Y.

Proof. Let f be ß-open, injective , then by Theorem 1.3. , $f^{-1}(B^{\circ-\circ}) \subset (f^{-1}(B))^{-}$. Since B $\subset$ Y is open , so $f^{-1}(\overline{B}^{o}) \subset (f^{-1}(B))^{-}$ Conversely , let V $\subset$ X be an open set , W = X – V and B = $(f(W))^{\circ}$. Then , $f^{-1}((f(W)^{\circ-\circ}) \subset (f^{-1}(f(W))^{\circ}))^{-} \subset (f^{-1}(f(W)))^{-} = \overline{W} = W$. Hence , $(f(W^{\circ-\circ})) \subset f(W)$, i.e. , f (W) is ß-closed and so, f is ß-open .

Definition 3.1. A mapping f : X $\to$ Y is called thickening. [14], if for every open set U $\subset$ X , f (U) is everywhere dense .

Now, it is clear that thickening implies ß-openness, but the converse may be false as the following example illustrates.

Example 3.1. Let X = {a, b, c} with topology T_X ={X , $\emptyset$, {a} , {a,b}} , and Y = {1 , 2 , 3 , 4} with T_Y = {Y , $\emptyset$,{1} , {2} ,{1, 2}} Then the mapping f : X → Y defined by f(a) = 2 , f(b) = 4 and f(c) = 1 , is ß-open but not thickening .

Definition 3.2. A mapping f: X →Y is called almost open in the sense of Singal and Signal, (briefly, a.o.s) if the image of each regularly open set is open.

The following theorem investigates the relationship between a.o.s, and ß-open mappings.

Theorem 3.2. A pre-continuous and a.o.s mapping is ß-open.

Proof. Let A $\subset$ X be open set , then $\overline{A}^{\circ}$ is regular open and f($\overline{A}^{\circ}$) is open in Y. Since f is pre-continuous , then by Theorem 1.1. [8], f(A) $\subset$ f($\overline{A}^{\circ}$) = $(f(\overline{A}^{\circ}))^{\circ} \subset (f(\overline{A}))^{\circ} = (f(A^{\circ -}))^{\circ} \subset (f(A))^{-\circ -}$. Therefore, f is ß-open.

Lemma 3.1. In a space X , G $\cap$ $\overline{A} \subset (G \cap A)^{-}$, for every A $\subset$ X , and G is an open subset of X .

Theorem 3.3. Let f : X → Y be a mapping and for each S $\subset$ Y , let f_S : f^{-1}(S) → S be the mapping which agrees with f on f^{-1}(S). If f is ß-open and S is open and closed in Y, then f_S is ß-open.

Proof. If U is an open subset of f^{-1}(S) , then there is an open subset V of X such that U = V $\cap f^{-1}$(S) , and therefore f_S(U) = f(V) $\cap$ S $\subset (f(V))^{-\circ -} \cap S = ((f(V))^{-} \cap S)^{\circ -}$. Now by the above lemma , S $\cap$ f(V) $\subset (S \cap f(V))^{-\circ -}$, so S$\cap$ f(V) is ß-open in Y. Since S is open, then S $\cap$ f(V) is ß-open is S and this that f_S is ß-open .

B-HOMEOMORPHISM.

The following lemma is very useful in the sequal.

Lemma 4.1. If f : X → Y is open , then

$$f^{-1}(\overline{A}) \subset (f^{-1}(A))^{-} .$$

Proof. Let A $\subset$ Y , W = Y – A , then f^{-1}(W) $\subset$ X . Since f is open , f(f^{-1} (W)$)^{\circ} \subset$ (f f^{-1}(W)$)^{\circ} \subset W^{\circ}$. So ($f^{-1}(W))^{\circ} \subset f^{-1}(W^{\circ})$ which implies $(f^{-1}(Y - A))^{\circ} \subset f^{-1}((Y - A)^{\circ})$.
Thus X – $\overline{f^{-1}(A)} \subset f^{-1}$(Y – $\overline{A}$)) = X – $f^{-1}(\overline{A})$. Therefore, $f^{-1}(\overline{A}) \subset \overline{f^{-1}(A)}$.

Now we give the following theorem.

Theorem 4.1. If f: X → Y is continuous and open , then f is M-ß-continuous .

Proof. If A$\in$ ß(y) , then f^{-1}(A) $\subset f^{-1}(\overline{A}^{\circ -}) = (f^{-1}(\overline{A}^{\circ}))^{-} \subset (f^{-1}(\overline{A}))^{\circ -} = (f^{-1}(A))^{-\circ -}$. Hence $f^{-1}(A)$ is ß-open subset of X and so f is M-ß-continuous .

Theorem 4.2. If f : X → Y is ß-continuous and $f^{-1}(\overline{V}) \subset (f^{-1}(V))^{-}$ for each ß-open set V$\subset$ Y . Then f is M- ß-continuous.

Proof. Let V $\in$ ß(y). Since f is ß-continuous, then f^{-1} (V) $\subset$ $f^{-1}(\overline{V}^{\circ -}) = (f^{-1}(\overline{V}^{\circ}))^{-} \subset (f^{-1}(\overline{V}^{\circ})^{-})^{- \circ -} = (f^{-1}(\overline{V}^{\circ}))^{- \circ -} \subset (f^{-1}(\overline{V}))^{- \circ -} = (f^{-1}(V))^{- \circ -}$. Hence f is M-ß-continuous.

Theorem 4.3. The bicontinuous image of a ß-open set of a space X is ß-open .

Proof. Consider f : X $\rightarrow$ Y is continuous and open mapping . Let A $\in$ ß(X) , then A $\subset A^{- \circ -}$, and so f(A) $\subset f(\overline{A}^{\circ -}) \subset (f(\overline{A}^{\circ}))^{-} \subset (f(\overline{A}))^{\circ -} \subset (f(A))^{- \circ -}$. Therefore , f(A) $\in$ ß(Y) .

Definition 4.1. A mapping f : X $\rightarrow$ Y which is M-ß-continuous and M-ß-open is called bi-M-ß-continuous. Considering the above definition, we give the following theorem.

Theorem 4.4. If f: X $\rightarrow$Y is continuous and open, then f is a bi-M-ß-continuous .
Proof. See Theorem 4.1, and 4.3.

Corollary 4.1. Each homeomorphism is a ß-homeomorphism.

Proof. It follows directly from Theorem 4.4.

The following example shows that the converse of the above corollary may be not true .

Example 4.1. Let X = Y = {a, b, c} with two topologies T_X = { X , $\emptyset$, {a} , {a, b}} , and T_Y = {Y , $\emptyset$,{a} , {a, b} , {a, c}}. Then the identity mapping i : X $\rightarrow$ Y is a ß-homeomorphism, but not homeomorphism .

The following example illustrates that continuity or M- ß-continuity need not be open.

Example 4.2. Let X = {a, b, c}, and consider the topologies T* = {X , $\emptyset$, {a} , {a, b}} , T ={X , $\emptyset$,{a} , {a, b} , {a, c}} on X . It is clear that ß (X , T) = ß (X, T*) . Let f : (X , T) $\rightarrow$ (X ,T*) be the identity mapping , then f is continuous and M- ß-continuous , but not open .

B-TOPOLOGICAL PROPERTIES.

Definition 5.1. A property, which is preserved under a ß-homeomorphism is called a ß-topological property.

Theorem 5.1. A ß-topological property is a topological property.

Proof. This result follows directly, since by Corollary 4.1. , every homeomorphism is ß-homeomorphism .

Now, it is clear from Example 4.1. that T_3, T_4 , T_5 , semi-regular, regular, completely regular, normal, completely normal and compact fail to be ß-topological properties .
Furthermore, the following example shows that T_0 , T_1 , T_2 and T_2' are not ß-topological properties .

Example 5.1. Let X be the countable set with two topologies : t_1 , the countable discrete topology , and t_2 , the indiscrete topology . Then the identity mapping i : (X , t_1) $\rightarrow$ (X , t_2) is a ß-homeomorphism , since every subset of (X , t_1) is ß-open and also , every subset in (X , t_2) is ß-open . But the indiscrete space (X , t_2) is not a T_0 , (T_1, T_2, T_2')-space .

Lemma 5.1. [9]. If f : X $\rightarrow$ Y is a ß-homeomorphism , then

i) ß-cl(f(A)) = f(ß-cl(A)) , for every A $\subset$ X .
ii) ß-int(f(A)) = f(ß-int(A)) , for each A $\subset$ X .

Now we give the following theorem .

Theorem 5.2. If' f : X $\rightarrow$ Y is a ß-homeomorphism , and A$\subset$ X is nowhere dense , then f(A) is nowhere dense in Y .

Proof. Since A is nowhere dense in X , then by Theorem, ß-int(ß- cl(A)) = $\emptyset$. Thus , by Lemma 5.1.,

$$ß\text{-int}(ß\text{-cl}(f(A))) = ß\text{-int}(f(ß\text{-cl}(A))) = f(ß\text{-int}(ß\text{-cl}(A))) = f(\emptyset) = \emptyset .$$

Therefore , f(A) is nowhere dense in Y .

We see from the above theorem that nowhere dense is a B-topological property.

Theorem 5.3. The property that a topological space be of the first category is a ß-topological property .

Proof. Consider f : X $\rightarrow$ Y is a ß-homeomorphism . Let X be of the first category , then $X = \bigcup_{i=1}^{\infty} A_i$, where A_i is nowhere dense for each i $\in$ I . Thus, Y = f ($\bigcup_{i=1}^{\infty} A_i$) = $\bigcup_{i=1}^{\infty} f(A_i)$. Since A_i is nowhere dense for each i $\in$ I , then By Theorem 5.2. , $f(A_i)$ is nowhere dense for each i $\in$ I and therefore Y is of the first category .

Corollary 5.1. The property that a topological space be of the second category is a ß-topological property.

Theorem 5.4. ß-homeomorphism is an equivalence relation between topological spaces .

Proof. Reflexivity and symmetry are immediate and transitivity follows from the fact that the composition of two M-ß-continuous (M-ß-open) mapping is M-ß-continuous (M-ß-open) mapping.

ß-topological classes. If X is a set of points T(X) denote the collection of all topological spaces , which , have X as their set of points . Now, we give the following , definition .

Definition 5.2. If X is a set of points and if (X , t) and (X , t*) are two elements of T(X) , then (X , t) is ß-correspondent to (X , t*) if and only if ß(X , t) = ß (X , t*) .

Now , we introduce the following theorem .

Theorem 5.5. ß-correspondent is an equivalence relation on the collection T(X).

Proof. (i) Clearly (X, t) is ß-correspondent to itself for any topology t on X.

(ii) Symmetry follows from symmetry of set equality.

(iii) Transitivity follows from transitivity of set equality.

Thus the collection T(X) of topological spaces is partitioned into equivalent classes.

CONCLUSION

The connections between these sorts of non-continuous mappings and various kinds of mappings as almost – continuous sense of Signal and Signal (a. o. s.), weakly continuous, thickening, almost – continuous sense of Signal and Signal (a. o . s.) are investigated. The concept of a ß- topological property, for example the property of being a topological space be of the first category is one of ß- topological properties.

REFERENCES

[1] Abd El-Monsef, ME and AM kozea, 1985.some generalized forms of Compactness, Delta J.sci. 9(2).
[2] Biswas, N., 1969. On some mappings in topological space, Bull. Calcutta Math. Soc ., 61: 127 -135 .

[3] Boum, J.D., 1964. Elements of point set topology, prentice Hall, London.
[4] Hermann, R.A., 1979.C- convergence, pros. Amer. Math. Soc. 75(2): 311-317.
[5] Husain, J., 1977. Topology and Maps, Plenum Press, New York.
[6] Iseki, K., 1961. Generalizations of the notion of compactness, Acad. de la Republique Polpulaire Raumaine, 7: 31-63 .
[7] Levine, N., 1963. Semi-open sets and semi continuity in topological spaces, Amer. Math. Monthely, 70: 36-41.
[8] Liu, C T., 1968. Absolutely closed spaces, Trans Amer. Math. Soc., 130: 86-104.
[9] Mashhour, A., S. Hasanian, I. A. El-Deeb, S N, 1982. A note on semi-continuity and pre-continuity, Indian J. pure Appl. Math. 13(10): 1119-1123.
[10] Mashhour, A., S. Hasanian and I.A. El-Deeb, 1983. α- continuous and α- open mappings, Acta. Math. Hungar, 41: 213-218.
[11] Poter, J., J. Thomas, 1969. On H-Closed and minimal Hausdorff spaces, Trans., Amer. Math. Soc., 138:159-170.
[12] Scarhorough, C.T. and A.H. Stone, 1979. Product of nearly compact spaces Trans, Amer. Math. Soc., 75: 311-317.
[13] Singal, M. J. and A. Rani, 1968. Almost-continuous mappings, Yokohama, Math. J., 16:63 -73.
[14] Wilansky, A., 1967. Topics in functional analysis, Springer, Berlin.