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Simulation of LR Fuzzy Random Variables with Normal Distribution

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Abstract: This work deals with the simulation of fuzzy random variables (FRVs) which is essential in modelling of various real situations. There are many situations in which random data are reported with uncertainty due to the constraints of measurement systems and/or human subjectivity. In such situations, only FRVs can describe the subject and cover uncertainty due to the fuzzy vagueness as well as randomness. In this study, based on the concept of support functions, the general method of FRVs simulation is extended especially for triangular/trapezoidal and arcuate shapes in LR family with Normal distribution.

Key words: Simulation · LR-FRV · Gaussian FRV · Support function

INTRODUCTION

In many real-life problems the available observations are not real-valued but rather imprecise valued. Uncertain information can take on many different forms. There is uncertainty which arises from ignorance, from various classes of randomness, from the inability to perform adequate measurements, from lack of knowledge or from vagueness, like the fuzziness inherent in our natural language [1-3].

In recent decades, fuzzy sets are increasingly used in various contexts like optimization, image processing, learning, decision-making, data analysis, engineering and control systems [3, 4]. There are many complex phenomena in which classical logic and probability theory are not able to describe and analyse them properly [5]. Further, there exist some problems in which they do not necessarily need to have exact solutions but rather an approximate and fast solution can be useful in making preliminary decisions [3]. In all of the mentioned problems, fuzzy variables characterize fuzzy uncertainty very well.

However, there are many situations which consist of two kinds of uncertainties and therefore there is a need of combining the classical set theory and statistical models along with fuzzy sets theory [6-9]. For example, consider the situation that we face with randomness which comes from selecting the random sample instead of considering the whole population. Further second kind of uncertainty is related to the fuzziness of the data and/or the parameters. Fuzzy random variable is vehicle to modelling conditions in the presence of them [9, 10]. The defined probability distribution models the stochastic variability of observations, while the defined membership function models the vagueness in a system [8]. In this way, the concept of an FRV that extends the classical definition of a random variable was introduced by Féron [11]. Kwakernaak [12] conceptualized an FRV as a vague perception of a crisp but unobservable RV that taking fuzzy value instead of real values. Further Puri and Ralescu [2] conceptualized the FRV as a fuzzification of a random set, whose values are fuzzy subsets of $\mathbb{R}^p$ or, more generally, of a Banach space. Later on and sometimes independently, other variants were proposed by Kruse and Meyer [13] and Diamond and Kloeden [14]. Krätschmer [15] surveyed all of these definitions and proposed a unified approach. In all of these works, an FRV is defined as a function which assigns a fuzzy subset to each possible output of a random experiment that intend to model situations that combine fuzziness and randomness [15-19].

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Despite of excellences of fuzzy sets in describing imprecise data, fuzzy arithmetic is accompanying with difficulties and complexities that may decrease utility of them. Computational efficiency is a particular importance when fuzzy set theory is used to solve real problems [5]. Concerning these problems, Dubois and Prade [20] introduced a flexible parametric family of membership functions which are called LR family. Using this family, computational efficiency increases without limiting the generality beyond acceptable limits [5].

Therefore simulation of FRVs is necessary and essential in checking the results validity and comparing and modelling of various real situations. Colubi *et al*. [7] simulated different types of FRVs to get some conclusions concerning fuzzy-valued random variables. They considered only fuzzy random variables with a range that contained in some restrictive parametric classes. González-Rodríguez *et al*. [4] represented two different approaches based on the concept of support functions to simulate FRVs. The first one makes use of techniques for simulating Hilbert space-valued random elements and then projected on the cone of all fuzzy sets. Several empirical results showed the practicability of this approach is limited. Another approach imitates the representation of every element of a separable Hilbert space in terms of an orthonormal basis directly on the space of fuzzy sets.

The approach which is based on support functions has a very good control on the simulation and the generated FRVs exhibit a shape similar to their expected value. However, simulation of the LR-FRV under a special probability distribution has not been investigated in earlier works. LR fuzzy random variables introduce distinct pattern and easy fuzzy arithmetic properties with respect to the other types. In addition the most of phenomena follow the Normal distribution or can be easily transformed into it. Therefore in this study the simulation of the LR-FRV class with Normal distribution is practically performed and some numerical examples are included to show the behaviour and validity of it.

In section 2, the preliminaries of the fuzzy sets are briefly mentioned. The concept of FRVs and respective theorems and definitions are presented in section 3. A simulation method based on the support function and alpha cuts for LR-FRVs with Normal distribution is explained in section 4. Section 5 illustrates the proposed approach with numerical examples and finally, concluding remarks make up the last section.

Preliminaries: Throughout the whole paper, the family $F(\mathbb{R})$ of all fuzzy sets in $\mathbb{R}$, defined by

$$F(\mathbb{R}) = \left\{ \tilde{B} : \mathbb{R} \to [0,1] \,\middle|\, \tilde{B}_\alpha \in K_C(\mathbb{R}) \quad \forall\, \alpha \in [0,1] \right\}$$

is considered. Thereby $K_c(\mathbb{R})$ denotes the class of all nonempty compact convex subsets of $\mathbb{R}$. The Set $B_\alpha = \left\{ x \in \mathbb{R} \,\middle|\, \tilde{B}(x) \geq \alpha \right\}$ is the α–cut or α–level or worthy set of $\tilde{B}$ [21] and the interval $\tilde{B}_0 = cl\left(supp(\tilde{B})\right)$, is called the support of the fuzzy set $\tilde{B}$, The family of all fuzzy sets with compact support is denoted by $F_C(\mathbb{R})$.

Support function of a crisp set $A \in \mathbb{R}^p$ is defined as $S_A(u) = \{\sup\langle u, a\rangle | a \in A,\, u \in S^{p-1}\}$, where $\langle .,. \rangle$ is the inner product of the Euclidean space $\mathbb{R}^p$ and S^{p-1} is the $(p-1)$-dimensional unit sphere of $\mathbb{R}^p$. Now whereas α–level of the fuzzy set $\tilde{B}$ is a crisp set [21], so the support function of the convex α–cut of the fuzzy set $\tilde{B}$ for any fixed $u \in S^{p-1}$ and for all $\alpha \in [0,1]$ is defined as $S_{\tilde{B}}(u,\alpha) = \left\{ \sup\langle u, b\rangle \,\middle|\, b \in B_\alpha,\, u \in S^{p-1} \right\}$.

A metric on the set of all normal compact convex fuzzy subsets of $\mathbb{R}^p$, is defined by the L_2-metric on the space of lebesgue integrable functions as bellow

$$d_2(\tilde{A},\tilde{B}) = \left\| S_{\tilde{A}} - S_{\tilde{B}} \right\|_2 = \left(p \times \int_0^1 \int_{S^{p-1}} \left| S_{\tilde{A}}(u,\alpha) - S_{\tilde{B}}(u,\alpha) \right|^2 L(du)\, d\alpha \right)^{\frac{1}{2}}$$

where L denotes the normed lebesgue measure on unit sphere S^{p-1}. Also a norm is defined as

$$\|\tilde{A}\|_2 = \|S_{\tilde{A}}\|_2 = \left(p. \int_0^1 \int_{S^{p-1}} \left| S_{\tilde{A}}(u,\alpha) \right|^2 L(du)\, d\alpha \right)^{\frac{1}{2}}.$$

Fuzzy Random Variables: Fuzzy random variables are measurable functions from a probability space to the set of fuzzy variables in Banach space that conceptualized by Puri and Ralescu [2]. In this work, simulation of FVRs in real space is studied. Therefore definition of FRV in $\mathbb{R}$ by González-Rodríguez *et al*. [4] is considered.

Definition 1: A fuzzy random variable $\tilde{x}$ on a probability space (Ω,A,P) is a $F(\mathbb{R})$ valued mapping on Ω which is measurable with respect to the metric d_2 (i.e. measurable with respect to the Borel σ-field generated by d_2). Which for all $\alpha \in [0,1]$, the mapping X_α: $\Omega \rightarrow K_C(\mathbb{R})$ is a compact random set [4].

Nevertheless, in order to make the simulation process practical, the following proposition is used:

Proposition 1: Every $\tilde{X} \in F_C(\mathbb{R})$ is a fuzzy random variable, if and only if $\underline{\tilde{X}}_\alpha$ and $\overline{\tilde{X}}_\alpha$ are random variables for all $\alpha \in [0,1]$, [10, 22].

Where $\underline{\tilde{X}}_\alpha$ and $\overline{\tilde{X}}_\alpha$ are denoted the minimum and maximum of $\tilde{X}_\alpha$, respectively.

Dubois and Prade [20] presented fuzzy numbers in a new format namely LR fuzzy numbers. In this way, the LR-FRVs are the most common class of FRVs, which are very varied in their shapes and have great utility to use, because of distinct pattern and easy fuzzy arithmetic. Furthermore, this family allows us to express every FRV in terms of three or four random variables, namely, "the center", "1/2 the core length", "the left spread" and "the right spread" [23]. Thus, we study the general approach proposed by González-Rodríguez *et al*. [4] in the format of LR-FRV class as a special case that is very applicable. Definition of LR-FRV is presented as below [8, 24]:

Definition 2: An LR-FRV denoted by $\tilde{X}_{LR} = (c,s,l,r)_{LR}$, has a membership function as:

$$\mu_{\tilde{X}_{LR}}(x) = \begin{cases} 0 & x < c-s-l \\ L(\frac{c-s-x}{l}) & c-s-l \le x < c-s \\ 1 & c-s \le x < c+s \\ R(\frac{x-s-c}{r}) & c+s \le x < c+s+r \\ 0 & c+s+r \le x \end{cases} \qquad , x \in \mathbb{R}$$

where c (the central value) is a square integrable random variable, s(1/2 the core length), l and r(the left and right spread, respectively), are three positive square integrable random variables and c, s, l and r are independent. L and R are fixed functions that satisfied:

- L and R are left-continuous and non-increasing functions and are mapping $\mathbb{R}^+ \rightarrow [0,1]$.
- $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.
- $\forall\, x > 0 \quad L(x)<1, \quad R(x)<1$.
- $\forall\, x > 0 \quad L(x)>1, \quad R(x)>1$.

The expectation of an FRV is shown uniquely with a fuzzy set [17], while the variance of an FRV is determined in two different approaches which are crisp and fuzzy set valued [25, 26]. In this study, a crisp variance for the FRV is assumed. When it is attempted to simulate the FRVs for using in a special problem, such as validation of results or empirical checking for theoretical subjects, often a particular distribution with known mean and variance is considered. Therefore following theorems are presented to use in the simulation process.

Theorem 1: The expected value of the LR-FRV, $\tilde{X}_{LR} = (c,s,l,r)_{LR}$, is the LR fuzzy set and is denoted by [24]:

$$E(\tilde{X}_{LR}) = (E(c), E(s), E(l), E(r))_{LR} \tag{1}$$

Theorem 2: Let $\tilde{X}_{LR} = (c,s,l,r)_{LR}$ is the LR-FRV, then the crisp variance of $\tilde{X}_{LR}$ follows:

$$Var(\tilde{X}_{LR}) = Var(c) + \|A_I\|_2^2 Var(s) + \|A_R\|_2^2 Var(r) + \|A_L\|_2^2 Var(l) \tag{2}$$

where $\|A_L\|_2^2 = \frac{1}{2}\int_0^1 (L^{(-1)}(\alpha))^2 d\alpha$ and $\|A_R\|_2^2 = \frac{1}{2}\int_0^1 (R^{(-1)}(\alpha))^2 d\alpha$ [24].

Proposition 2: The variance of a trapezoidal LR-FRV, as a special case, when L(x)= R(x)=1-x is determined by

$$Var(\tilde{X}_{LR}) = Var(c) + Var(s) + \frac{1}{6}Var(r) + \frac{1}{6}Var(l) \tag{3}$$

and if $P(s = 0) = 1$, then $Var(\tilde{X}_{LR}) = Var(m) + \frac{1}{6}Var(r) + \frac{1}{6}Var(l)$

Proof: It is clear that $\|A_I\|_2^2 = 1$. If $L(x) = R(x) = 1 - x$, then $L^{-1}(\alpha) = R^{-1}(\alpha) = 1 - \alpha$ and $\|A_L\|_2^2 = \frac{1}{2}\int_0^1 (L^{(-1)}(\alpha))^2 d\alpha = \frac{1}{2}\int_0^1 (\alpha - 1)^2 d\alpha = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ as well as $\|A_R\|_2^2 = \frac{1}{6}$. Thus Eq. (3) is obtained. ■

Gaussian LR-FRV is considered in this study, therefore its definition presented in the following. Puri and Ralescu [27] introduced the concept of a Gaussian FRV with value in the set of all Lipschitz fuzzy numbers on $\mathbb{R}^p$. Definition of Gaussian FRV is [27]:

Definition 3: Fuzzy random variable $\tilde{X} : \Omega \to F_C(\mathbb{R}^p)$ is Normal, if $s_{\tilde{X}}$ is a Gaussian random element of $C\left([0,1] \times S^{p-1}\right)$, the Banach space of continues functions.

On the other hands, Wu [22] defined probability distribution for an FRV with fuzzy parameters as bellow:

Definition 4: Let X be a random variable having distribution with parameters $\theta_1,...,\theta_n$ and $\tilde{X}$ be a fuzzy random variable. Then X_α^l and X_α^u are random variables for all $\alpha \in [0,1]$. Fuzzy random variable $\tilde{X}$ is said to have the same distribution as X with fuzzy parameters $\tilde{\theta}_1,...,\tilde{\theta}_n$, if for all $\alpha \in [0,1]$, X_α^l and X_α^u has a same distribution as X with parameters $\tilde{\theta}_{1\alpha}^l,...,\tilde{\theta}_{n\alpha}^l$ and $\tilde{\theta}_{1\alpha}^u,...,\tilde{\theta}_{n\alpha}^u$ respectively.

As a result of above definition, $\tilde{X}$ is normally distributed with fuzzy parameters $\tilde{\mu}$ and $\tilde{\sigma}^2$ if and only if $X_\alpha^l \sim N\left(\mu_\alpha^l, \sigma^{2}{}_\alpha^l\right)$ and $X_\alpha^u \sim N\left(\mu_\alpha^u, \sigma^{2}{}_\alpha^u\right)$ for all $\alpha \in [0,1]$, [10, 22]. If a crisp variance is assumed, then $\tilde{X} \sim N\left(\tilde{\mu}, \sigma^2\right)$, if and only if $X_\alpha^l \sim N\left(\mu_\alpha^l, \sigma^2\right)$ and $X_\alpha^u \sim N\left(\mu_\alpha^u, \sigma^2\right)$ for all $\alpha \in [0,1]$.

Claim 1: Definition 3 in $\mathbb{R}$ and definition 4 in the case of Normal distribution are equivalent.

Proof: Def. 3 to Def. 4

Let $\tilde{X} \in \mathbb{R}$ is an FRV which $s_{\tilde{X}}$ is Normal. The unit sphere in $\mathbb{R}$ follows $S^{1-1} = S^0 = \{-1,1\}$. In this case, the Support function of $\tilde{X} \in \mathbb{R}$ is simplified as:

$$S_{\tilde{X}}(u,\alpha) = \begin{cases} \sup\left(-\tilde{X}_\alpha\right) = \min\ \tilde{X}_\alpha = \tilde{X}_\alpha^l & ;u = -1 \\ \sup\left(\tilde{X}_\alpha\right) = \max\ \tilde{X}_\alpha = \tilde{X}_\alpha^u & ;u = 1 \end{cases}$$

for all $\alpha \in [0,1]$. Thus $\tilde{X}_\alpha^l$ and $\tilde{X}_\alpha^u$ are Normal for all $\alpha \in [0,1]$.

Def. 4 to Def. 3

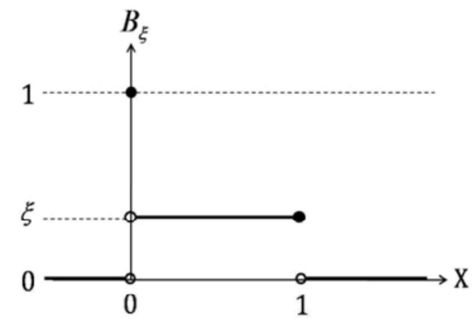

Fig. 1: Fuzzy set B_ξ

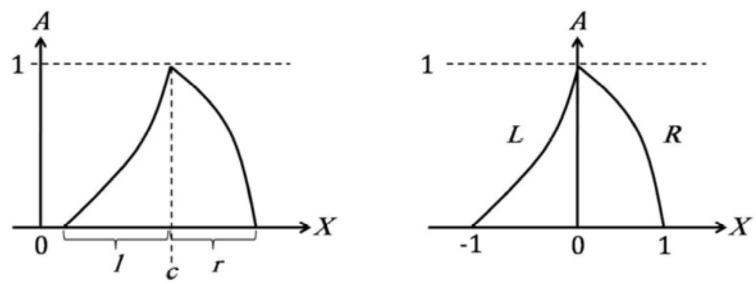

Fig. 2: Components of a fuzzy set (a) the crisp numbers *c*, *l* and *r*, (b) the *L* and *R* functions.

Let $\tilde{X} \in \mathbb{R}$ is a Normal FRV which $\tilde{X}_\alpha^l$ and $\tilde{X}_\alpha^u$ are Normal for all $\alpha \in [0,1]$. Thus $S_{\tilde{X}}$ is Normal ▪.

Using proposition 1 and equivalency of two recent definitions, the method of simulation of LR-FRV with Normal distribution, is presented in the next section.

Simulation Methods

Simulation of LR-FRVs: In order to describe the approach exactly, it is necessary to theoretically analyze the existence of a (countable) "basis" for the family $F_C(\mathbb{R})$. It can be shown there exists an uncountable family, *G*, that can be regarded as a generator for $F_C(\mathbb{R})$, [4]. $G = \{B_\xi: \xi \in [0,1]\}$ is a family of FRVs that for every realization of the random variable $\xi \in [0,1]$, the result is the fuzzy set B_ξ (Fig. 1), that its formula has been given by $B_\xi = I_{\{0\}}(x) + \xi.I_{[0,1]}(x)$.

Whereas every fuzzy set $\tilde{A} \in F_C(\mathbb{R})$, can be characterized uniquely with the family of their α-levels [4], so an LR-FRV can be generated with simulation of some arbitrary α-cuts. For this purpose, the approach is begun with a simple (non-stochastic) decomposition, then it is approximated these components in the selected α-cuts. As is mentioned in the previous section, any fuzzy set $\tilde{A} \in F_C(\mathbb{R})$ can be written as [4, 24]:

$$\tilde{A} = c + l.L + r.R \tag{4}$$

where *c* is the central point of the purposed fuzzy set core and *l* and *r* are the left and right spread of it. Further $c \in \mathbb{R}$, $l, r \in \mathbb{R}^+$ are fixed scalar. *L* and *R* are the fixed fuzzy sets were introduced in Def. 2. The crisp numbers *c, l, r* and the *L* and *R* shape functions are shown in Fig. 2.

It can be shown [4]:

$$L = E(-B_{1-X_l}) \quad , \quad R = E(B_{1-X_r}) \tag{5}$$

where X_l and X_r are two [0,1]-valued random variables. $1 - X_l$ and $1 - X_r$ are used as the random variable ξ which was mentioned in the *G* family. Further *E*(.) denotes the Aumann expectation of FRVs. Approximation of the shape functions is subsequently performed.

The [0, 1] interval should be divided to n parts, in which the value of n is equal to number of the α-cuts. The components of the given fuzzy set (function) will be simulated in these levels, where $0 = \alpha_1 < \alpha_2 <...<\alpha_n = 1$. These amounts will be regarded as the realizations of random variables X_l and X_r. Then with considering cumulative distribution functions of X_l and X_r, following amounts are calculated as below:

$$p_1^l = F_{X_l}(\alpha_1) \quad , \quad p_i^l = F_{X_l}(\alpha_i) - F_{X_l}(\alpha_{i-1}) \quad i = 2,...,n$$
$$p_1^r = F_{X_r}(\alpha_1) \quad , \quad p_i^r = F_{X_r}(\alpha_i) - F_{X_r}(\alpha_{i-1}) \quad i = 2,...,n \tag{6}$$

Using the p_i^l 's and p_i^r 's, a proper approximate for Eq. (5) can be written as follows:

$$L \approx E(-B_{1-\alpha_i}) = \sum_{i=1}^{n} (-B_{1-\alpha_i})\, p_i^l \quad , \quad R \approx E(B_{1-\alpha_i}) = \sum_{i=1}^{n} (B_{1-\alpha_i})\, p_i^r \tag{7}$$

In this way, the connection of the shapes functions and distributions of random variable X_l and X_r is expressed in the following proposition which was presented by González-Rodríguez *et al*. [4] without any proof.

Proposition 3: For every realization of X_l and X_r, it holds:

$$F_{X_l}(\xi) = \max(-L_{1-\xi}) \quad , \quad F_{X_r}(\xi) = \max(R_{1-\xi}) \tag{8}$$

Proof: The proof is presented by limit concept for the R function. A similar manner can be used for the L function. By placing Eq. (6) into the Eq. (5), it can be written:

$$R = E(B_{1-X_r}) \approx E(B_{1-\alpha_i}) = \sum_{i=1}^{n} (B_{1-\alpha_i})\, p_i^r$$
$$= (B_{1-\alpha_1}) F_{X_r}(\alpha_1) + \sum_{i=2}^{n} (B_{1-\alpha_i})(F_{X_r}(\alpha_i) - F_{X_r}(\alpha_{i-1}))$$

Every fuzzy set is specified with alpha levels uniquely; moreover every alpha cut of a fuzzy set in $\mathbb{R}$ is a crisp interval that is identified with two its end points. Thus the end points of the R function alpha levels are written as following: (It should be noticed that $(B_\alpha) = 0$ and $\max(B_\alpha) = 1$ for all $\alpha \in [0,1]$)

$$\min(R)_{1-\alpha_i} = \underbrace{\min(B_{1-\alpha_1})}_{0} F_{X_r}(\alpha_1) + \sum_{j=2}^{i} \underbrace{\min(B_{1-\alpha_j})}_{0} (F_{X_r}(\alpha_j) - F_{X_r}(\alpha_{j-1})) = 0$$

$$\max(R)_{1-\alpha_i} = \underbrace{\max(B_{1-\alpha_1})}_{1} F_{X_r}(\alpha_1) + \sum_{j=2}^{i} \underbrace{\max(B_{1-\alpha_j})}_{1} (F_{X_r}(\alpha_j) - F_{X_r}(\alpha_{j-1}))$$
$$= F_{X_r}(\alpha_1) + F_{X_r}(\alpha_2) - F_{X_r}(\alpha_1) + ... + F_{X_r}(\alpha_{i-1}) + F_{X_r}(\alpha_i) - F_{X_r}(\alpha_{i-1}) = F_{X_r}(\alpha_i)$$

By tending the value of n to the extreme, the exact solution is obtained. Thus, the amount of cumulative distribution functions of X_r, in the point ξ, is equal to maximum of R function α-cut in the level $1-\xi$. In this way, the connection between shape functions with the distribution for random variables X_l and X_r is revealed. ■

For example, triangular or trapezoidal FRV are used more than other shapes. The suitable distributions for X_l and X_r to simulate a triangular or trapezoidal FRV are found in following:

Claim 2: If the random variables X_l and X_r to be generated from the Uniform distribution on [0,1] then the simulated fuzzy sets are triangular or trapezoidal FRV in $\mathbb{R}$.

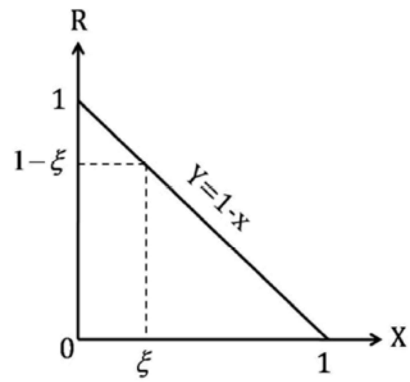

Fig. 3: $R(x)$=1-x.

Proof: Trapezoidal and triangular FRV are a special cases of LR-FRV family that occur when $L(x) = R(x) = 1 - x$. The random variables X_l and X_r must be generated from probability distribution which satisfies Eq. (8). If $X_r = \xi$ to be observed, then $F_{X_r}(\xi) = \max(R_{1-\xi}) = \xi$, (Fig. (3)). It means X_r must be selected from a distributions in which $F_{X_r}(\xi) = \xi$, for every $\xi \in [0,1]$ Thus in this case, the Uniform distribution on [0,1] is suitable and $X_l, X_r \overset{iid}{\sim} U[0,1]$.

■

Consequently, according to Eq. (5) the fuzzy set $\tilde{A}$ can be rewritten as follows:

$$\tilde{A} = c + l.E(-B_{1-X_l}) + r.E(B_{1-X_r}) \tag{9}$$

And in the same manner, it can be approximated by:

$$\tilde{A}_n = c + \sum_{i=1}^{n}(-B_{1-\alpha_i})c_i^l + \sum_{i=1}^{n}(B_{1-\alpha_i})c_i^r \tag{10}$$

where

$$c_i^l = l.p_i^l \geq 0 \quad , \quad c_i^r = r.p_i^r \geq 0 \quad , \quad i = 1,...,n \tag{11}$$

Typically, in a specific problem, simulation of an FRV with particular distribution and special expectation is considered. Thus the expected value of the purposed FRV is selected as the fixed fuzzy set $\tilde{A}$. Therefore the L and R functions and the crisp numbers c, l and r are revealed. According to Eq. (10) the expectation of the approximating fuzzy set $\tilde{A}_n$ is given by [4]:

$$E(\tilde{A}_n) = E(C) + \sum_{i=1}^{n}(-B_{1-\alpha_i})E(C_i^l) + \sum_{i=1}^{n}(B_{1-\alpha_i})E(C_i^r)$$

where $C: \Omega \to \mathbb{R}$ and $C_i^l, C_i^r : \Omega \to [0,\infty)$, are random variables in which $E(C) = c$, $E(C_i^l) = c_i^l$ and $E(C_i^r) = c_i^r$, for $i=1,...,n$. Based on this idea, the simulation of an LR-FRV is proposed to do according to the following steps:

Step 1: Select the number $n \in \mathbb{N}$ sufficiently large and determine the α_i's where $0 = \alpha_i <...< \alpha_n = 1$.

Step 2: With considering the cumulative distribution functions of X_l and X_r, calculate the p_i^l's and the p_i^r's values according to Eq. (6) and the c_i^l's and the c_i^r's according to Eq. (11).

Step 3: Consider a (2n+1)-dimensional random vector

$$\underline{Y} = \left(C, C_1^l, ..., C_n^l, C_1^r, ..., C_n^r\right) : \Omega \to \mathbb{R} \times [0,\infty)^{2n}$$

of coefficients for the 'approximating' given FRV as random perturbation of $\left(c, c_1^l, ..., c_n^l, c_1^r, ..., c_n^r\right)$ in such a way that

$$E\left(C, C_1^l, ..., C_n^l, C_1^r, ..., C_n^r\right) = \left(c, c_1^l, ..., c_n^l, c_1^r, ..., c_n^r\right)$$

holds. Afterwards, generate one realization of the random variable C, in such a way $E(C) = c$. Further, generate n realizations of each random variables Z_l and Z_r, such that $E(Z_l) = E(Z_r) = 1$. Consequently consider the random variables bellow:

$$C_i^l = z_i^l.c_i^l \qquad , \qquad C_i^r = z_i^r.c_i^r$$

now, the random vector $\underline{Y}$, is ready to construct a fuzzy set as the realization of preceding FRV.

Step 4: Repeat three previous steps for $j = 1,...,m$ and generate samples $\underline{Y}_1,...,\underline{Y}_m$ of $\underline{Y}$, then construct fuzzy sets $\tilde{A}_j$ by Eq. (12):

$$\left(\tilde{A}_j\right)_{1-\alpha_1} = \left[C - C_1^l, C + C_1^r\right], \left(\tilde{A}_j\right)_{1-\alpha_i} = \left(\tilde{A}_j\right)_{1-\alpha_{i-1}} + \left[-C_i^l, C_i^r\right]; i = 2,...,n \tag{12}$$

Eq. (12) represents the amount of the fuzzy set $\tilde{A}_j$ in its α-cuts, so the fuzzy set $\tilde{A}_j$ appears with connect these points to each other. Clearly, the larger number $n \in \mathbb{N}$ will result in better simulation. Although for the triangular or trapezoidal LR-FRV that are most applicable, $n=2$ can be sufficed for accuracy of the simulation process.

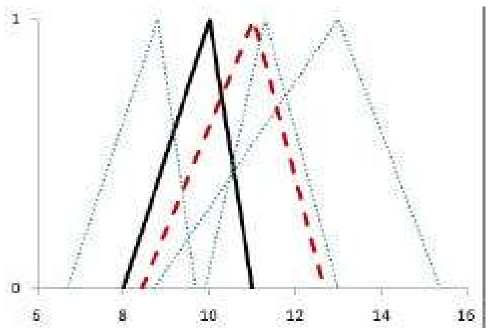

Fig. 4: Fuzzy expected value (fat line), simulated realizations (dotted line) and fuzzy sample means (dash line), n=3.

Simulation of an LR-FRV with Normal Distribution: Simulation of the Gaussian LR-FRV is presented in the following:

Claim 3: If the random variables C, Z_l and Z_r, to be generated from a Normal distribution, then the simulated fuzzy numbers are distributed as a Gaussian FRV.

Proof: According to Def. 3, if the support function of FRV to be normally distributed then the simulated fuzzy numbers are normally distributed too. Moreover, according to equivalency of definition 3 and definition 4, the simulated fuzzy numbers have a Normal distribution if for all $\alpha \in [0,1]$, $\tilde{X}_\alpha^l$ and $\tilde{X}_\alpha^u$ be Normal. In the other words, for simulating a Gaussian FRV, the central value and the left and right spread of the LR-fuzzy sets must be generated from Normal distribution. Therefore $s_{\tilde{X}}$ will be Normal and so $\tilde{X}$ will be normally distributed. ■

As was mentioned, in order to simulate the FRVs for using in particular problem, often a special distribution with known mean and variance is considered. Let the simulation of FRVs with distribution as $N\left(\tilde{A},\sigma^2\right)$ is necessary, where $\tilde{A}=c+l.L+r.R$. For this purpose, the random variable C must be generated from $N\left(c,\sigma_c^2\right)$, where c is the central point of the expected value core of the purposed FRV. Moreover, the random variables Z_l and Z_r must be generated from $N\left(1,\sigma_l^2\right)$ and $N\left(1,\sigma_r^2\right)$, respectively. According to Eq. (3) the variance of simulated fuzzy numbers, σ^2, is equal to $\sigma_c^2+\frac{\sigma_l^2+\sigma_r^2}{6}$. In this way, the central value and the left and right spread of the simulated fuzzy numbers are normally distributed. As a result, the support functions of simulated fuzzy numbers follow the Normal distribution and according to Def. 3, the simulated fuzzy numbers follow $N\left(\tilde{A},\sigma^2\right)$.

In the other case, let the simulation of FRVs with distribution as $N\left(\tilde{A},\sigma^2\right)$ be desirable, where $\tilde{A}=c+s.I+l.L+r.R$ is a trapezoidal FRV. The simulation process is followed as before unless the random variables Z_l and Z_r must be generated as $N\left(1,\sigma_s^2\right)$ for i=1; and $N\left(1,\sigma_l^2\right)$ and $N\left(1,\sigma_r^2\right)$ for i=2. According to Eq. (3) the variance of simulated fuzzy numbers, σ^2, is equal to $\sigma_c^2+\sigma_s^2+\frac{\sigma_l^2+\sigma_r^2}{6}$. In this way, the central value and the left and right spread of the simulated fuzzy numbers for all $\alpha \in [0,1]$ are normally distributed. Thereby the simulated fuzzy numbers follow $N\left(\tilde{A},\sigma^2\right)$.

Numerical Examples: In order to show the treatment of this approach some numerical examples are included. The simulation method for triangular FRV with Normal distribution is illustrated in example 1. Moreover examples 2 and 3 check the validity of the method using d_2 metric. Examples use the simulated data which have been generated by SAS/IML programming [28-33].

Example 1: The simulation of the LR-FRV is desired with a distribution as $N\left(\tilde{A},4.5\right)$, where $\tilde{A}=(10,2,1)_{LR}$ is the triangular LR-FRV. According to Eq. (6) and Eq. (11) the coefficients of step 2 are calculated. In addition $\sigma_c^2=4$, $\sigma_l^2=2$ and $\sigma_r^2=1$ are considered. Whereas, in this case c=10, l=2 and r=1, so the random variables C, Z_l and Z_r, are generated from N(10, 4), N(1,2) and N(1,1) respectively. Thus the simulated fuzzy numbers are Gaussian LR-FRVs whit fuzzy mean $\tilde{A}$ and crisp variance 4.5. Table 1 shows the simulation results for three times.

Fuzzy set $\tilde{A}$ is shown in Fig. 4 as well as simulated realizations and fuzzy sample mean of them.

Example 2: Accuracy of this method is studied using d_2 metric. In this case, the fuzzy set $\tilde{A}$ in the example 1 is simulated for some values of n, from 3 to 100'000 times. The fuzzy set $\tilde{A}$ and the fuzzy sample mean of simulated fuzzy realizations are compared in each stage and are shown in Fig. 5. The difference is too small for large samples. Hence the distance between population parameter and the sample mean is determined based on d_2 metric to discern the difference clearly. These amounts are available in Table 2.

As it is seen, the sample mean is matched to the population mean from n=5000 with negligible difference.

Table 1: Simulation of Gaussian LR FRV with fuzzy mean $\tilde{A}$ and crisp variance 4.5.

	i	$C_i^l = z_i^l . c_i^l$	C	$C_i^r = z_i^r . c_i^r$	$(c, l, r)_{LR}=(c-l, c, c+r)_T$
j=1	1	0	8.8	0	$(8.8, 2.1, 0.9)_{LR}=(6.7, 8.8, 9.7)_T$
	2	2.1		0.9	
j=2	1	0	13	0	$(13, 4.3, 2.4)_{LR}=(8.7, 13, 15.4)_T$
	2	4.3		2.4	
j=3	1	0	11.3	0	$(11.3, 1.4, 1.7)_{LR}=(9.9, 11.3, 13)_T$
	2	1.4		1.7	

Table 2: The distance between fuzzy population parameter and fuzzy sample mean based on d_2 metric

n	3	10	50	100	200	500	1000	5000	20'000	100'000
d_2	1.109	0.966	0.118	0.269	0.132	0.137	0.12	0.064	0.096	0.057

Table 3: The distance between fuzzy population parameter and fuzzy sample mean based on d_2 metric

n	5	10	25	50	100	500	1000	10000
d_2	0.992	0.404	0.937	0.2	0.163	0.082	0.135	0.04

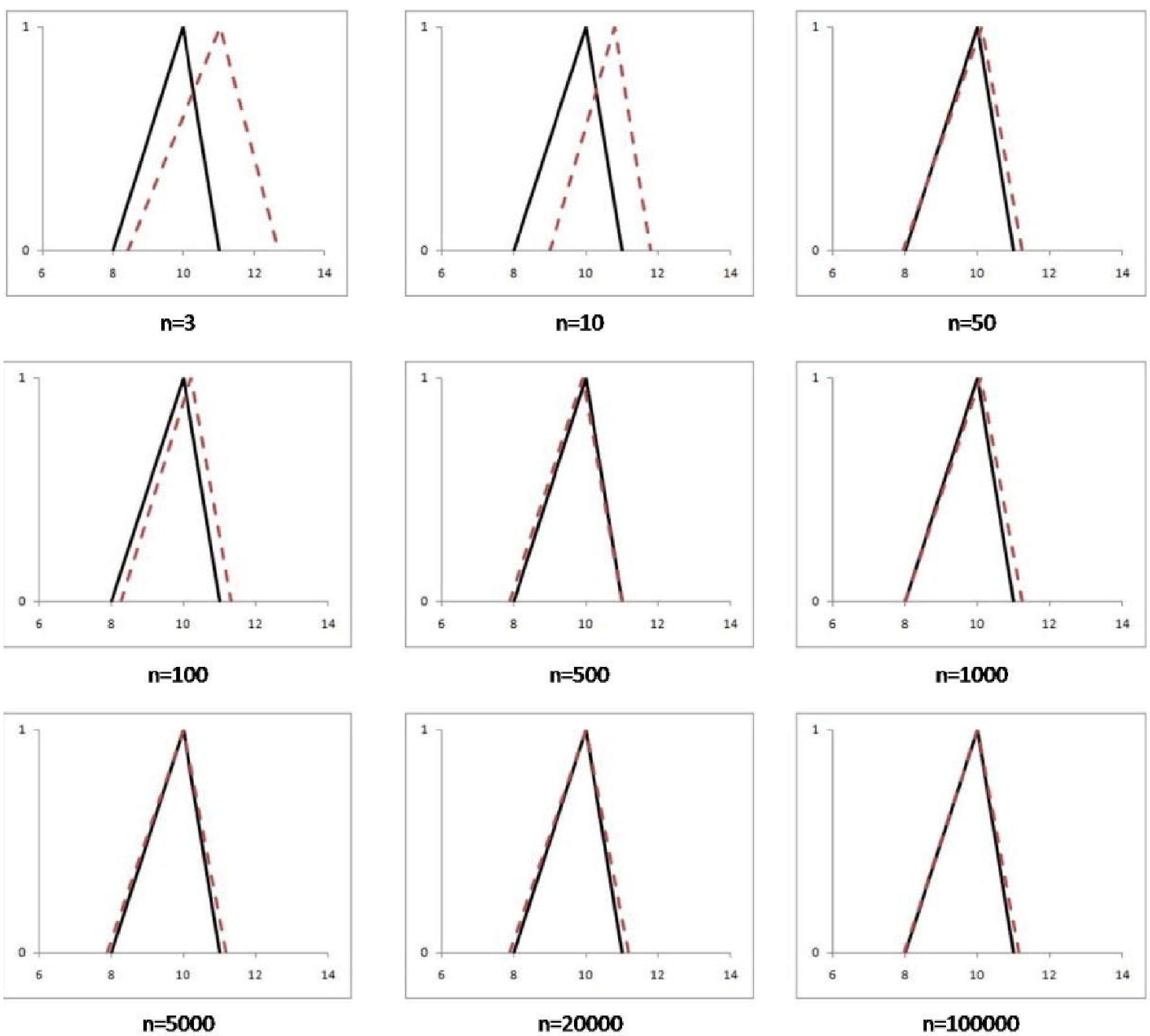

Fig. 5: Comparison between fuzzy expected value (fat line) and fuzzy sample mean (dash line)

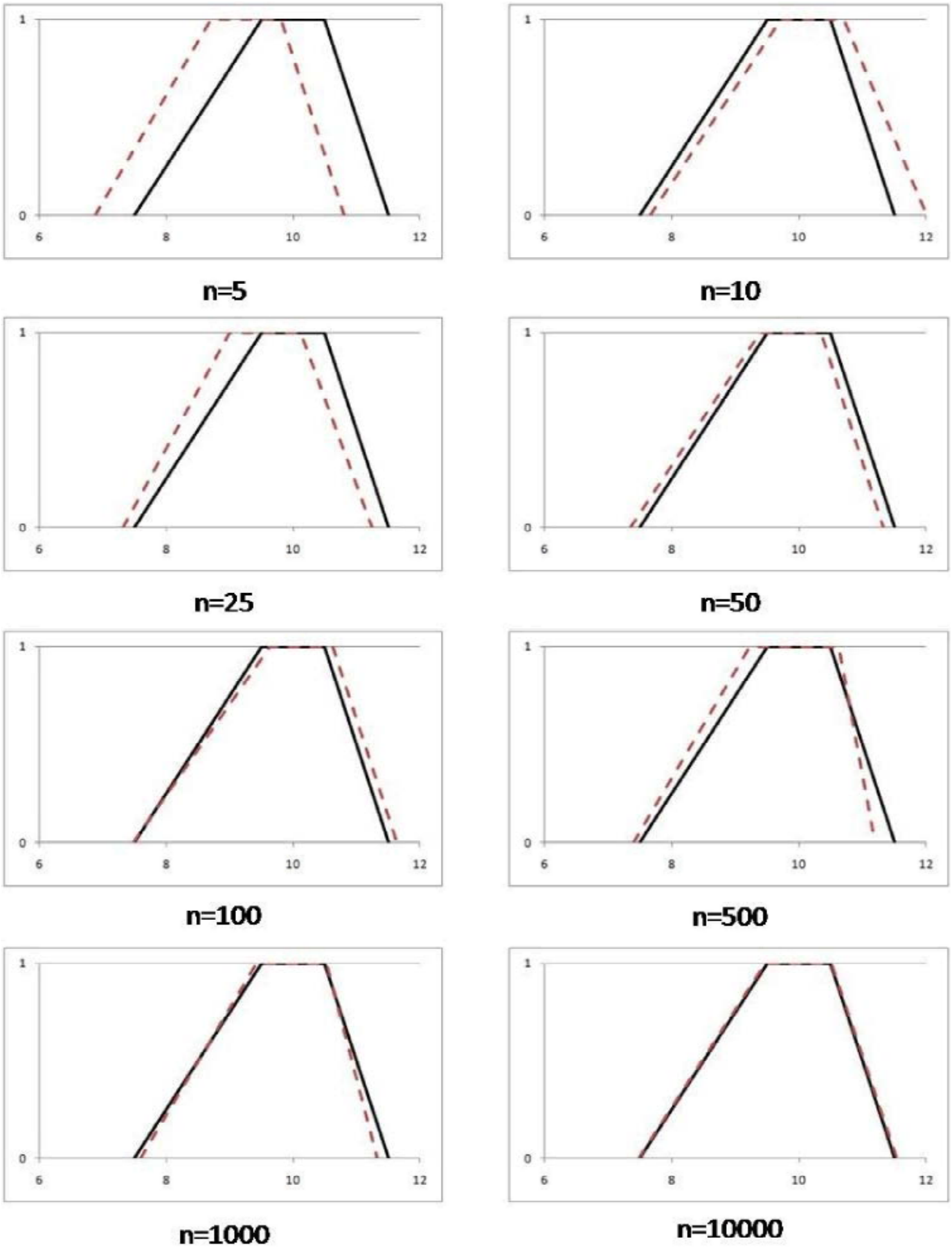

Fig. 6: Comparison between fuzzy expected value (fat line) and fuzzy sample mean (dash line)

Example 3: The simulation of the LR-FRV with a distribution as $N\left(\tilde{A},4.37\right)$ is considered where $\tilde{A}=(10,0.5,2,1)_{LR}$ is of the trapezoidal FRV type. It should be noticed that the core of a trapezoidal FRV is an interval instead of a scalar. Hence the values of p_1^l and p_1^r are equal to 1. In addition $\sigma_c^2=4$, $\sigma_s^2=0.2$, $\sigma_l^2=0.5$ and $\sigma_r^2=0.5$ are desired. The simulation process is performed for some values of *n*, from 5 to 10'000 times. Figure 6 shows the comparison between the fuzzy parameter $\tilde{A}$ and the fuzzy sample mean of simulated fuzzy realizations. Based on d_2 metric, the distance between population parameter and the sample mean is determined in order to clearly discern of the difference. These amounts are available in Table 4.

In both recent examples, the sample means moves towards the population parameter by growing the sample size. Thereby the method generates the valid data according to the goal.

Concluding Remarks: Simulation of LR-FRVs with Normal distribution was studied because of their applications. Typically in LR family, triangular or trapezoidal FRV are used more than other shapes. Suitable distributions for random variables X_l and X_r was found, till the simulated fuzzy numbers to be had triangular/trapezoidal and arcuate shapes.

Some numerical examples were included to indicate the treatment of this approach, using the simulated data which have been generated by SAS/IML programming. The simulation methods for triangular and trapezoidal FRV with Normal distribution are discussed. Further in order to study the accuracy and validity of the method, the fuzzy parameter of the population mean was compared with the fuzzy sample mean in some different sample size. For this purpose, d_2 metric was used and it was shown the sample mean moves towards the population parameter with growing the sample size. It means that the simulation method operates accurately and generates the valid data according to the target.

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