Middle-East Journal of Scientific Research 11 (8): 1171-1175, 2012
ISSN 1990-9233

DOI: 10.5829/idosi.mejsr.2012.11.08.1734

Some Results in Infinite System with a Toeplitz Matrix

Massoud Mashreghi

Institute of Mathematics NAS, Armenia

Abstract: In this paper an infinite system of a Toeplitz matrix was considered and some results are demonstrated as partially new solution for the system. A survey was conducted on infinite system based on Toeplitz matrix computation. A unique solution of Markov chains is presented with some advance technique of canonical solution. Markov chains can be exploited as the basis of stable algorithms, for the computation of the solution of the nonlinear matrix. Computational integrals are carried out with operations involved in infinite Toeplitz matrix system. In this work, the integral solution of Markov chains of nonlinear matrix equations are presented.

Key word: Toeplitz matrix . Markoff chain . Factorization equation . Canonical solution . Yengibarian's system . Karlin's theorem

INTRODUCTION

Let us define the following system of equation are applied in the theory of Markoff chains [1-4].

$$f_i = g_i + \sum_{j=-\infty}^{\infty} k_{i-j} f_j, i = 0, \pm 1, \pm 2, \ldots \qquad (1)$$

It is assumed here that matrix $K = (k_{i-j})_{i,j=-\infty}^{\infty}$ has the following properties:

$$k_j \geq 0, j = 0, \pm 1, \pm 2, \ldots, \sum_{j=-\infty}^{\infty} k_j = 1 \qquad (2)$$

$$\sum_{j=-\infty}^{\infty} |j| k_j < +\infty, \upsilon \overset{def}{=} \sum_{j=-\infty}^{\infty} j k_j > 0 \qquad (3)$$

The following facts and theorems are defined by Karlin and Schwab-Felisch [5, 6]. These findings having an important role in the theoretical approaches and equations as stated above.

Theorem I: Let us state the following conditions for the summation of inequality:

a. $\sum_{j=-\infty}^{\infty} |g_j| < +\infty$

b. The greatest common divisor of the indices which $k_j > 0$ is equal to 1; that was realized in the system of equation (1) according to the conditions given by equations (2) and (3). If this equation has a bounded solution $f = (f_j)_{j=-\infty}^{\infty}$, then there exists $\lim_{j \to \pm\infty} f_j$ and if $\lim_{j \to -\infty} f_j = 0$, then the following limit is stated:

$$\lim_{j \to +\infty} f_i = \frac{1}{\upsilon} \sum_{j=-\infty}^{\infty} g_i$$

On the other hand, the following theorem is proved.

Theorem II: Let us state the following system:

$$\tilde{f}_i = \tilde{g}_i + \sum_{j=0}^{i} k_{i-j} \tilde{f}_j, i = 0, 1, 2, \ldots \qquad (4)$$

such that $\sum_{j=-\infty}^{\infty} |\tilde{g}_j| < +\infty$ and sequence $\{k_j\}_{j=-\infty}^{\infty}$ satisfies the conditions stated in equation (2) and also the specified condition (b) of the Theorem I which $k_j = 0$ for j<0, if $\tilde{f} = \{\tilde{f}_j\}_{j=0}^{\infty}$ is a bounded solution of the system equation (4) then

$$\lim_{j \to +\infty} \tilde{f}_i = \frac{1}{\upsilon_+} \sum_{j=0}^{\infty} g_i \qquad (5)$$

Where

$$\upsilon_+ \overset{def}{=} \sum_{j=1}^{\infty} j k_i$$

Corresponding Author: Massoud Mashreghi, Institute of Mathematics NAS, Armenia

Remarks: It should be noted that the system of equation (4) is obtained from equation (1) which is based on condition if and only if $k_j = g_j = 0$ for j<0.

Let us say $\{k_j\}_{j=-\infty}^{\infty}$ be a two-sided sequence of real numbers which satisfying the condition

$$\mu \overset{def}{=} \sum_{j=-\infty}^{\infty} |k_j| < +\infty$$

This sequence generates Toeplitz matrix $K = (k_{i-j})_{i,j=0}^{\infty}$.

Let us define Ω as a class of considered matrices and Ω^+ and Ω^- be subclasses of the upper and lower triangle matrices:

$$A = (a_{i-j})_{i,j=0}^{\infty} \in \Omega^+; \text{ if } A \in \Omega \text{ and } a_j = 0 \text{ for } j<0;$$

$$B = (b_{j-i})_{i,j=0}^{\infty} \in \Omega^-; \text{ if } B \in \Omega \text{ and } b_j = 0 \text{ for } j>0;$$

The matrix $K \in \Omega$ generates a linear bounded operator acting in e^+ ($e^+ = l_p$; $p \geq 1$; m^+ or c^+ where m^+ and c^+ are sets of all matrices $(k_i)_{i=0}^{\infty}$ that are bounded and convergent to zero, respectively) by the matrix multiplication, with the following relation:

$$\|K\|_{e^+} \leq \mu$$

The classes $\Omega^{\pm}$ are algebras relative to multiplication while Ω is not a relation such an algebra. The important algebraic property of the matrices from Ω^+ and Ω^- is that if $A \in \Omega^+$ and $B \in \Omega^-$ then $BA \in \Omega$.

The following factorization problem has been investigated and are reported in the literatures [3, 7-9]. Let us define $K \in \Omega$. That is required to find $A \in \Omega^+$ and $B \in \Omega^-$ such that:

$$I - K = (I - B)(I - A) \tag{6}$$

where I represents an infinite unit matrix.

Equality equation (6) may be rewritten as the following non-linear algebraic system (the system is known as Engibarian's system) [3] relative to the $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$:

$$a_i = k_i + \sum_{j=0}^{\infty} b_j a_{i+j}$$

$$b_i = k_{-i} + \sum_{j=0}^{\infty} a_j b_{i+j} \tag{7}$$

$$a_0 + b_0 = k_0 + \sum_{j=0}^{\infty} a_j b_j, \ i = 1,2,...$$

The sequence $\{k_j\}_{j=-\infty}^{\infty}$ in (I) Also generates a Toeplitz (two-sided) matrix $\tilde{K}$ of the type $\tilde{K} = (k_{i-j})_{i,j=-\infty}^{\infty}$.

A similar factorization problem may be set forward for the matrices $\tilde{K}$. Let us define $\tilde{\Omega}$ to be a matrix class of the type $\tilde{K}$ and $\tilde{\Omega}^{\pm}$ be subclasses of the upper and lower triangle two-sidedly infinite matrices. That is required to find $\tilde{A} \in \tilde{\Omega}^+$ and $\tilde{B} \in \tilde{\Omega}^-$ such that:

$$\tilde{I} - \tilde{K} = (\tilde{I} - \tilde{B})(\tilde{I} - \tilde{A}) \tag{8}$$

whereas the term $\tilde{I}$ is an infinite two-sided unite matrix.

It appears that factorization equation (8) is reduced to the same system of equation (7), relative to a_j and b_j, where $\tilde{A} = (a_{i-j})_{i,j=-\infty}^{\infty}$ and $\tilde{B} = (b_{i-j})_{i,j=-\infty}^{\infty}$. with $a_j = b_j = 0$ for j<0.

We shall introduce certain facts from well defined literatures [10, 11], necessary for our further consideration. It is necessary to add the system of equation (7) with the relation $b_0 = \rho a_0$, $\rho \geq 0$. Then from equation (7) one can obtain the following relations:

$$a_i = k_i + \sum_{j=0}^{\infty} b_j a_{i+j}$$

$$b_i = k_{-i} + \sum_{j=0}^{\infty} a_j b_{i+j} \tag{9}$$

$$(1+\rho)a_0 = k_0 + \sum_{j=0}^{\infty} a_j b_j, \ i = 1,2,...$$

Concerning the matrices K and $\tilde{K}$, we shall assume that conditions stated in equation (2) are realized.

As stated in defined literature [7], it was considered the iterations $a^{(p)} = \{a_j^{(p)}\}_{j=0}^{\infty}$ and $b^{(p)} = \{b_j^{(p)}\}_{j=0}^{\infty}$ the solution (a,b) of the system of equations (9) or (8) are determined according to the following equalities

$$a_i^{(p+1)} = k_i + \sum_{j=0}^{\infty} b_j^{(p)} a_{i+j}^{(p)}$$

$$b_i^{(p+1)} = k_{-i} + \sum_{j=0}^{\infty} a_j^{(p)} b_{i+j}^{(p)} \tag{9}$$

$$(1+\rho)a_0^{(p+1)} = k_0 + \sum_{j=0}^{\infty} a_j^{(p)} b_j^{(p)}, i = 1,2,...$$

$$b_0^{(p)} = \rho a_0^{(p)}, \ i = 1,2,3,... \ ; \ p = 0,1,2,...$$

If conditions stated in equation (2) are realized, then the solution (a,b) where $a=\{a_j\}_{j=0}^{\infty}$ and $b=\{b_j\}_{j=0}^{\infty}$ and iterations demonstrated in equation (10) for all $\rho\in(0,\infty)$ will acquire the following properties; these facts and findings are reported in the literatures [7, 12] $0\le a_j^{(p)}, b_j^{(p)}\uparrow$ with respect to p for all j;

$$0\le\gamma_{\pm}^{(p)}\le 1, 0\le\gamma_{\pm}\le 1$$

where

$$\gamma_{+}^{(p)}=\sum_{j=0}^{\infty}a_i^{(p)},\ \gamma_{+}=\sum_{j=0}^{\infty}a_i,\ \gamma_{-}^{(p)}=\sum_{j=0}^{\infty}b_i^{(p)},\ \gamma_{-}=\sum_{j=0}^{\infty}b_i$$

(c) $(1-\gamma_{-})(1-\gamma_{+})=0$

The last equality means that at least one of the values $\gamma_{\pm}$ is equal to 1 if the conditions stated in equation (2) are realized.Which of these values is equal to 1 (and hence, which of the factors in equalities stated in equations (6) and (8) cannot be converted into e^{+} or e) is solved on the basis of the following theorem; also similar situations are reported in the literatures [3, 10, 13, 14].

Theorem III: Let us define the sequence $K=\{k_j\}_{j=-\infty}^{\infty}$ in the system of equation (9) satisfies the conditions given by equation (2) with $\upsilon_{\pm}\overset{\text{def}}{=}\sum_{j=1}^{\infty}jk_{\pm j}<+\infty$. Then

a. $\upsilon>0\Leftrightarrow\gamma_{+}=1,\gamma_{-}<1$;

b. $\upsilon<0\Leftrightarrow\gamma_{+}<1,\gamma_{-}=1$;

c. $\upsilon=0\Leftrightarrow\gamma_{\pm}=1$;

where

$$\upsilon\overset{\text{def}}{=}\upsilon_{+}-\upsilon_{-}=\sum_{j=-\infty}^{\infty}j\,k_j.$$

Problem statement: Let η be a finite or infinite one or two-side vector. We shall designate the set of indices with $\eta_j\neq0$ as G_η; and the greatest common divisor of the elements of G_η as d_η.

Since Toeplitz matrices $K=(k_{i-j})_{i,j=r}^{\infty}$; where r = 0 or r = -∞ are determined by one- sided or two-sided vectors; their sets G_k are similarly determined.

One has to know if greatest common divisor of the elements of G_k is d_k; what relation is between it and d_a or d_b and in addition to existence of the solution for equation (4) in the conservative case of equation (2) should be an important issue of the solution.

Solution: The present paper has focused on defining a solution to the last question in the following theorem (Theorem IV).

Let us define $K=(k_{i-j})_{i,j=r}^{\infty}$; where r = 0 or r = -∞ and $k_j\ge0$ for all j and

$$\sum_{j=1}^{\infty}k_j>0\ (\text{or}\ \sum_{j=1}^{\infty}k_{-j}>0) \tag{11}$$

Then $d_a = d_k$ (or $d_a = d_k$) where (a,b) is the solution of the system of equation (9).

Proof: At first, it should be noted that the set G_a is not empty. Actually, from equation (11) it has to follow that exist t>0 such that a_t>0. From equation (9) and from that numbers a_t and b_t are not negative; it follows that $a_t\ge k_t>0$.
Therefore, $t\in G_a$.

In our further considerations t will be fixed. We shall now prove that $d_a\le d_c$.

It is enough to prove that if there exist integer d≥1 and $p\in G_a$ such that p is not divided by d, then there may be found such that $q\in G_a$ that could not be divided by d.

Let us know that p = -s<0. In the next step select the number $r\in G_a$ such that d|r (in the opposite case if the chosen number r is not completely divided by d, then q may be taken equal to r). From system of equation (9) we have $b_s\ge k_{-s}>0$ and

$$b_n\ge b_{n+r}a_r,\ \text{for}\ \forall n\in N \tag{12}$$

Let us define s>r.

Assuming in equation (12) that n = s-r one can obtain $b_{s-r}\ge b_s a_r>0$.

Then assuming in equation (12) n = s-2r, n = s-3r,… in a sequence, we obtained b_{s-jr}>0 for all j; $j\in N, j=\frac{s}{r}$. t is chosen such that t = s-jr>0, 0<t<r. Let us define q = r-t. from equation (9) we had

$$a_q=a_{r-t}\ge a_r q_r>0$$

where q is not divided by d, as q = (j+1)r-s with r being undivided by d and s being undivided. We shall prove that any common divisor d of the set G_k is a common divisor for G_a and G_b. Actually, let us know that $k_{\pm1}=0$ if $i\not\equiv0(\text{mod } d)$.

Then from equation (10), by induction with respect to n, it is proved that equations $a_i^{(p)}=0, b_i^{(p)}=0$ which are true for i, where as $i\not\equiv0(\text{mod } d)$. Therefore, $a_i = 0$, $b_i = 0$.

Corollary: From the proven theorem we particularly obtained that if $d_k = 1$ and if the first condition in equation (11) is satisfied, then $d_a = 1$ with $a_0<1$. But if the second condition in equation (11) is realized and $d_k = 1$, then $d_b = 1$, $d_0 = 1$.

Theorem V: Let us define in equation (4) condition in equation (2) to be realized, where $k_0<1$ and $\tilde{g} = \{\tilde{g}_j\}_{j=0}^{\infty} \in l_1$; then there exists solution $\tilde{f} = \{\tilde{f}_j\}_{j=0}^{\infty}$ of this equation and $\tilde{f} \in m$.

Proof: Since equation (4) is a linear equation, it is enough to consider the case $\tilde{g} \geq 0$ (i.e. $\tilde{g}_j \geq 0$ for all $j \in N$). We shall consider iterations

$$\tilde{f}^{(p)} = \{\tilde{f}_j^{(p)}\}_{j=0}^{\infty}, p = 0,1,2,...$$

determined by the following equalities

$$\tilde{f}_i^{(p+1)} = \tilde{g}_i + \sum_{j=0}^{i} k_{i-j}\tilde{f}_j^{(p)}, \tilde{f}^{(0)} = \{0,0,0,...\}, p = 0,1,2,... \quad (13)$$

It is evident that $\tilde{f}_j^{(p)}$ increases with respect to p for all $j \in N$. Then for an arbitrary $r \in N$ from equation (13) we obtained:

$$\sum_{i=0}^{r} \tilde{f}_i^{(p)} \leq \sum_{i=0}^{r} g_i + \sum_{i=0}^{r}\sum_{j=0}^{i} k_{i-j}\tilde{f}_j^{(p)}$$

Taking into the account the $\mu = \sum_{j=0}^{\infty} k_j = 1$ and $k_0<1$, it follows that

$$\tilde{f}_r^{(p)} \leq \frac{1}{1-k_0}\sum_{j=0}^{\infty}\tilde{g}_i \quad (14)$$

Since the values $\tilde{f}_j^{(p)}$ monotonously increase with respect to p, from estimations of equation (14) it follows that there exist a limit $\tilde{f} = \{f_0, f_1, \ldots\} \in m$ of the iterations of equation (13), which will evidently satisfy equation (4) while

$$\tilde{f}_j^{(p)} \uparrow \tilde{f}_j \text{ and } \tilde{f}_j \leq \frac{i}{1-k_0}\|\tilde{g}\|_{l_1}$$

are true for all $j \in N$.

If, as a supplement to the last theorem conditions, there exists $\upsilon_+ \overset{\text{def}}{=} \sum_{j=1}^{\infty} jk_i$ for the sequence $k = \{k_j\}_{j=0}^{\infty}$ then from the Theorem II it follows the relation discussed in literatures [5, 15].

$$\lim_{i\to\infty}\tilde{f}_i = \frac{1}{\upsilon_+}\sum_{j=0}^{\infty} g_i \quad (15)$$

CONCLUSION

This paper presents a partially new solution for Toeplitz matrix system. Special type of operator was applied. Approximation functions are used; integral solution of convolution type was successfully considered. Canonical factorization equations were specifically applied.

REFERENCES

1. Bini, D. and F. Benedetto, 1990. A new preconditioner for the parallel solution of positive definite Toeplitz systems. ACM_symposium on Parallel algorithms and architectures, doi: 10.1145/97444.976881990. New York, NY, USA: ACM, pp: 220-223.
2. Bini, D.A., L. Gemignani and B. Meini, 2002. Computations with infinite Toeplitz matrices and polynomials. Linear Algebra and its Applications, 343: 21-61.
3. Engibaryan, N.B. and M. Mnatsakanyan, 1977. Linear algebraic systems with Toeplitz matrices. USSR Computational Mathematics and Mathematical Physics, 17 (5): 3-17.
4. Van Den Hof, P.M.J., S.C.P. Heuberger and J. Bokor, 1995. System identification with generalized orthonormal basis functions. Automatica, 31 (12): 1821-1834.
5. Karlîn, S., 1955. On the renewal equation. Pacific J. Math., 5: 229-257.
6. Schwab-Felisch, O., 2007. Umriss eines allgemeinen Begriffs des musikalischen Satzmodells. Zeitschrift der Gesellschaft for Musiktheorie, Vol: 4 (3).
7. Rudin, W., 1991. Functional analysis. International series in pure and applied mathematics. McGraw-Hill, New York.
8. Feller, W., 2008. An introduction to probability theory and its applications. John Wiley & Sons New York.
9. Stenflo, Ã., 2001. A note on a theorem of Karlin. Statistics & Probability Letters, 54 (2): 183-187.
10. Engibaryan, N.B. and A. Arutyunyan, 1975. Integral equations on the half-line with difference kernels and nonlinear functional equatons. Matematicheskii Sbornik, 139 (1): 35-58.

11. Engibaryan, N.B., 1998. Renewal theorems for a system of integral equations. Sbornik: Mathematics, 189: 1795.
12. Engibaryan, N.B., 1996. Convolution equations containing singular probability distributions. Izvestiya: Mathematics, 60 (2): 251-279.
13. Engibaryan, N.B., 2000. Setting and solving several factorization problems for integral operators. Sbornik: Mathematics, 191: 1809.
14. Meini, B., 1998. Solving M/G/l type Markov chains: Recent advances and applications. Stochastic Models 14 (1-2): 479-496.
15. Karlin, S. and H.M. Taylor, 1981. A second course in stochastic processes. Academic Press.