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Introduction to Green's Function and its Numerical Solution

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Abstract: This paper has provided a brief introduction to the use of Green's functions for solving Ordinary and Partial Differential Equations in different dimensions and for time-dependent and time independent problem. Under many-body theory, Green's functions is also used in physics, specifically in quantum field theory, electrodynamics and statistical field theory, to refer to various types of correlation functions, even those that do not fit the mathematical definition.
George Green (14 July 1793-31 May 1841) was largely self-taught British mathematical physicist who wrote "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism (Green, 1828)". The essay introduced several important concepts, among them a theorem similar to the modern Green's theorem, the idea of potential functions as currently used in physics and the concept of what are now called Green's functions. George Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as James Clerk Maxwell, William Thomson and others.
His work ran parallel to that of the great mathematician Gauss (potential theory).

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Key words: Green's function . boundary value problem . initial value problem . self-adjoint equations

INTERODUCTION

Definition: Consider first the one-dimensional problem of a thin rod occupying the interval (0, a) on the x axis [3-6, 9]. We obtain

$$\frac{d^2u}{dx^2} = f(x) \qquad 0 < x < 1$$
$$u(0) = \alpha \qquad u(1) = \beta \tag{1}$$

where $f(x)$ is the prescribed source density (per unit length of the rod) of heat and α, β are the prescribed end temperatures. The three quantities $\{f(x), \alpha, \beta\}$ are known collectively as the data for the problem. The data consists of the boundary data α, β and of the forcing function $f(x)$.

We shall be concerned not only with solving (1) for specific data but also with finding a suitable form for the solution that will exhibit its dependence on the data. Thus as we change the data our expression for the solution should remain useful. The feature of (1) that enables us to achieve this goal is its linearity, as reflected in the superposition principle:
If $u_1(x)$ is a solution for the data $\{f_1(x), \alpha_1, \beta_1\}$ and $u_2(x)$ for the data $\{f_2(x), \alpha_2, \beta_2\}$ then $Au_1(x) + Bu_2(x)$ is a solution for the data

$$\left\{Af_1(x) + Bf_2(x); A\alpha_1 + B\alpha_2, A\beta_1 + B\beta_2\right\}$$

In practice, the superposition principle permits us to decompose complicated data into possibly simpler parts, to solve each of the simpler boundary value problem and then to reassemble these solutions to find the solution of the original problem. One decomposition of the data which is often used is

$$\left\{f(x); \alpha, \beta\right\} = \left\{f(x); 0, 0\right\} + \left\{0; \alpha, \beta\right\}$$

The problem with data $\{f(x), 0, 0\}$ is an inhomogeneous equation with homogeneous boundary conditions, the problem with data $\{0; \alpha, \beta\}$ is a homogeneous equation with inhomogeneous boundary conditions.

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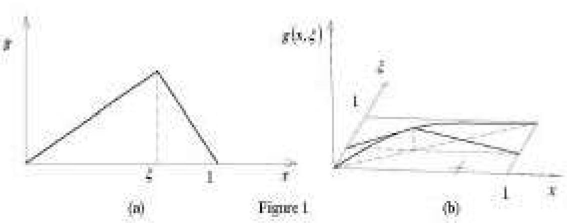

Figure 1

We show how the superposition principle or other methods lead to the following form for the solution of (1):

$$u(x) = \int_0^1 g(x,\xi) f(\xi)\,d\xi + (1-x)\alpha + x\beta \qquad (2)$$

Where Green's function $g(x, \xi)$ is a function of the real variables x and ξ defined on the square $0 \le x, \xi \le 1$ and is explicitly given by

$$g(x,\xi) = x_<(1 - x_>) = \begin{cases} x(1-\xi) & 0 < x < \xi \\ \xi(1-x) & \xi < x < 1 \end{cases} \qquad (3)$$

Here $x_<$ stands for the lesser of the two quantities x and ξ and $x_>$ for the greater of x and ξ.

In Fig. (1a) we picture Green's function as a function of x for fixed ξ and in Fig. (1b) as a function of x and ξ. Thus Fig. (1a) can be viewed as a cross section of the surface in Fig. (1b)

Now, Consider the problem of the forced, transverse vibrations of a taut string of length l. If the time-dependent parts of the solution are first removed by the usual separation-of-variation technique, we obtain the following differential equation containing the transverse displacement of the string, u, as unknown;

$$\frac{d^2u}{dx^2} + k^2 u = -f(x), 0 < x < l \qquad (4)$$

If the ends of the string are kept fixed, then this equation must be solved for u subject to the boundary conditions:

$$u(0) = u(l) = 0 \qquad (5)$$

To solve the boundary value problem posed by the ordinary second-order differential equation (4) and associated boundary conditions (5), we will employ the method of variation of parameters. That is, we will assume that a solution to the problem actually exists and that, furthermore, it has the precise form:

$$u(x) = A(x)\mathrm{Cos}(kx) + B(x)\mathrm{Sin}(kx)$$

Thus, we can write the solution of (2) in the form

$$u(x) = \frac{\cos(kx)}{k}\int_{c_1}^{x} f(\xi).\sin(k\xi)d\xi - \frac{\sin(kx)}{k}\int_{c_2}^{x} f(\xi)\cos(k\xi)d\xi \qquad (6)$$

where c_1 and c_2 are constants which must be so chosen as to ensure that the boundary conditions (5) are satisfied. We see that the solution (6) can be written in the form

$$\begin{aligned} u(x) &= \frac{1}{k}\int_0^x f(\xi)\sin(k(\xi - x))d\xi \\ &\quad - \frac{\sin(kx)}{k.\sin(kl)}\int_0^1 f(\xi)\sin(k(\xi - l))d\xi \\ &= \int_0^x f(\xi)\frac{\sin(k\xi).\sin(k(l-x))}{k.\sin(kl)}d\xi \\ &\quad + \int_x^1 f(\xi)\frac{\sin(kx).\sin(k(l-\xi))}{k.\sin(kl)}d\xi \end{aligned} \qquad (7)$$

$$= \int_0^1 f(\xi) g(x,\xi) d\xi \qquad (8)$$

Equation (8) is obtained from (7) by introducing the function $g(x, \xi)$ defined as

$$g(x,\xi) = \frac{\sin(k\xi).\sin(k(l-x))}{k.\sin(kl)} \qquad 0 \le \xi \le x$$

$$g(x,\xi) = \frac{\sin(kx).\sin(k(l-\xi))}{k.\sin(kl)} \qquad x < \xi \le l$$

This function, $g(x, \xi)$, is a two-point function of position known as the Green's function for the equation (4) and boundary conditions (5). Its existence is assured in this particular problem, provided that $\mathrm{Sin}(kl) \neq 0$.

In the general case, the differential equation we consider here is [1, 2, 4, 18]

$$L[y] + f(x) = 0, \ a < x < b \qquad (9)$$

First, we consider here the following boundary value problem with homogeneous differential equation:

$$\begin{cases} L[y] = 0 \\ y(a) = 0, y(b) = 0 \end{cases}$$

where

$$L[y] \equiv \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)y$$

We assume here that p(x), q(x) and *f*(x) are known and analytic on (a,b) and *f*(x) may be singular at x = a,b provided that the equation is defined properly.

Let y_1 and y_2 be fundamental solutions of L[y] = 0 which satisfy $y_1(a) = 0$ and $y_2(b) = 0$, respectively. Then the Green's function can be written

$$g(x,\xi)=\begin{cases} y_1(\xi)y_2(x), & \xi \le x \\ y_1(x)y_2(\xi), & x \le \xi \end{cases} \tag{10}$$

Under these preliminaries the solution of the boundary value problem of inhomogeneous equation (9) is expressed as follows:

$$u(x)=\int_a^b f(\xi)g(x,\xi)d\xi \tag{11}$$

Characterized of green's function: We have therefore characterized Green's function g(x, ξ) both physically and mathematically [1, 3], let us recapitulate what has been done so far.

1. Physically description. We chose to describe g in terms of heat conduction in a rod: g(x, ξ) is the temperature at x when the only source is a unit concentrated source at ξ, the ends being at 0 temperatures. It is also possible to interpret g as the transverse deflection of a string: g(x, ξ) is the deflection at x when the only load is a unit concentrated force at ξ, the ends being kept fixed on the x axis at $x = 0$ and $x = 1$.

2. Classical mathematical formulation. Green's function g(x, ξ) associated with (1) satisfies

$$\begin{aligned} &-\frac{d^2g}{dx^2}=0 \quad 0<x<\xi \ ,\ \xi<x<1 \\ &g(0,\xi)=g(1,\xi)=0 \\ &g \ \ continuous \ \ at \ \ x=\xi \\ &g'\big|_{x=\xi+} - g'\big|_{x=\xi-} = -1 \end{aligned} \tag{12}$$

In our third formulation we would like to consider (12) as a boundary value problem of the form (1) with specific data. The boundary data for g clearly vanishing, but what is the forcing function?

3. Delta function formulation.

Green's function g(x, ξ), of a linear differential operator L = L[x] acting on distributions over a subset of the Euclidean space $\mathfrak{R}^n$, at a point ξ, is any solution of

$$\begin{aligned} &Lg(x,\xi)=\delta(x-\xi) \quad 0<x<1,\ 0<\xi<1 \\ &g(0,\xi)=g(1,\xi)=0 \end{aligned} \tag{13}$$

where δ is the Dirac delta function.

This property of a Green's function can be exploited to solve differential equations of the form

$$Lu(x)=f(x)$$

If the kernel of L is non-trivial, then the Green's function is not unique. However, in practice, some combination of symmetry, boundary conditions and/or other externally imposed criteria will give a unique Green's function. Also, Green's functions in general are distributions, not necessarily proper functions

Green's functions are also a useful tool in solving wave equations, diffusion equations and in quantum mechanics, where the Green's function of the Hamiltonian is a key concept, with important links to the concept of density of states. As a side note, the Green's function as used in physics is usually defined with the opposite sign; that is

$$Lg(x,\xi)=-\delta(x-\xi)$$

This definition does not significantly change any of the properties of the Green's function

If the operator is translation invariant, that is when L has constant coefficients with respect to x, then the Green's function can be taken to be a convolution operator, that is

$$g(x,\xi)=\delta(x-\xi)$$

In this case, the Green's function is the same as the impulse response of linear time-invariant system theory

APPLICATIONS OF GREEN'S FUNCTION

The green's function for initial-value problems

Definition of the green's function: Choose $x = 0$ at the initial point [1, 2] and write the ICs as

$$\Psi(0)=\alpha,\ \Psi'(0)=\beta \tag{14}$$

The problem is to find a solution of

$$L\Psi(x)=f(x)$$

subject to (1), valid for all $x \ge 0$, for arbitrary $f(x)$.

The strategy is to satisfy (14) with an appropriate complementary function $\Phi_c(L\Phi_c=0)$ and to define a Green's function so that it delivers a particular integral that does not again upset the ICs. Thus we require

$$\Psi=\Phi_c+\Psi_p \tag{15}$$

$$\Phi_c(0)=\alpha,\ \Phi_c'(0)=\beta \tag{16}$$

$$\Psi_p(0) = 0,\ \Psi'_p(0) = 0 \tag{17}$$

In the special case of homogeneous ICs, i.e. $\alpha = 0 = \beta$, one can set $\Phi_c = 0$, i.e. one needs no complementary function at all.
We write the particular integral as

$$\Psi_p(x) = \int_0^\infty g(x,\xi) f(\xi) d\xi \tag{18}$$

where g obeys (13) (like all Green's functions); in order to guarantee (17) irrespective of *f*, we demand in addition

$$g(0,\xi) = 0,\ g_x(0,\xi) = 0 \tag{19}$$

Clearly, for the present we are considering g(x, ξ) as a function of x, with ξ merely a parameter.

Construction of green's function: g(x, ξ) is defined by (13) and (19). To construct it, observe that it obeys $Lg = 0$ for all x expect $x = \xi$. For $x < \xi$ and $x > \xi$, g can therefore be expressed as a linear combination of any pair Φ_1, Φ_2 of linearly independent solution of $L\Phi = 0$, though generally a different combination in the two regions. The pair Φ_1, Φ_2 can be chosen freely, without any reference to Φ_c (though of course Φ_c is necessarily expressible as a linear combination of them).

We deal separately with the two regions $x<\xi$, $x>\xi$ and then match the solutions across $x = \xi$.
In the region $0 \le x < \xi$,

$$g(x,\xi) = 0 \qquad (x < \xi) \tag{20}$$

In the region $\xi < x$, write

$$g(x,\xi) = A\Phi_1(x) + B\Phi_2(x) \qquad (\xi < x) \tag{21}$$

The solutions (20, 21) are matched across $x = \xi$ as described in Section (2.1).

In our present IVP, (20) shows that g(x, ξ) vanishes for all $x<\xi$; so consequently does $\partial g(x,\xi)/\partial x$. In particular, this is still true at $x = \xi$; therefore the continuity and jump conditions reduce to

$$g(\xi^+,\xi) = 0,\ \frac{\partial}{\partial x} g(x,\xi)\bigg|_{x=\xi^+} = -1 \tag{22}$$

It remains only to determine A and B satisfying (22), They give, respectively,

$$A\Phi_1(\xi) + B\Phi_2(\xi) = 0$$

and

$$A\Phi'_1(\xi) + B\Phi'_2(\xi) = -1$$

or in other words

$$\begin{bmatrix} \Phi_1(\xi) & \Phi_2(\xi) \\ \Phi'_1(\xi) & \Phi'_2(\xi) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{23}$$

These are two simultaneous, linear, inhomogeneous algebraic equations for the two unknown A and B. thus there exists a unique solution provided only that

$$\det[\ ] = W\{\Phi_1(\xi), \Phi_2(\xi)\} \equiv W(\xi) \neq 0$$

which is so, simply because Φ_1, Φ_2 are linearly independently. The solution is

$$A = \frac{\Phi_2(\xi)}{w(\xi)},\ B = -\frac{\Phi_1(\xi)}{w(\xi)} \tag{24}$$

We amalgamate (20) and (21) into

$$g(x,\xi) = H(x,\xi)\{A\Phi_1(x) + B\Phi_1(x)\}$$

substitute for A and B from (24) and write the result explicitly:

$$g(x,\xi) = H(x-\xi).\frac{\Phi_2(\xi)\Phi_1(x) - \Phi_1(\xi)\Phi_2(x)}{w(\xi)}$$

(Be careful to distinguish primes (on ξ) from derivatives (Φ').)
Thus the solution to our IVP reads

$$\Psi(x) = \Phi_c(x) + \int_0^\infty H(x-\xi) g(x,\xi) f(\xi) d\xi$$

Or

$$\Psi(x) = \Phi_c(x) + \int_0^x g(x,\xi) f(\xi) d\xi$$

The green's function for boundary-value problems: For BVPs, unlike IVPs, a Green's function may not exist, either because the problem has no solution, or because the solution in not unique.

For simplicity [1], in the present section we confine ourselves to homogeneous BC's

$$\Psi(x_1) = 0 = \Psi(x_2) \tag{25}$$

The problem is to solve, in the range $x_1 \le x \le x_2$, $L\Psi(x) = f(x)$ subject to (25), for arbitrary $f(x)$. It will appear presently that the existence and nature of the solution depend critically on whether the associated homogeneous problem

$$L\Phi = 0 \tag{26}$$

subject to the same (homogeneous) BC's has a solution. We shall distinguish between the general case, where (26) has no solution and the special case, where it has. The strategy is to look for Ψ in the form

$$\Psi(x) = \int_{x_1}^{x_2} g(x,\xi) f(\xi) d\xi \tag{27}$$

where

$$\begin{aligned} &Lg(x,\xi) = \delta(x-\xi) \\ &g(x_1,\xi) = 0 \quad , \quad g(x_2,\xi) = 0 \end{aligned} \tag{28}$$

This g obeys the same differential equation as did the (different) g in the IVP's, but now it satisfies the BC's (5b,c) inspired by (25).

We construct g from the solution of the homogeneous equation (26), which is satisfies by g for $x<\xi$ and $x>\xi$. Let Φ_1 be a solution that obeys the left-hand BC; let Φ_2 be a solution that obeys the right-hand BC:

$$L\Phi_{1,2} = 0,\ \Phi_1(x_1) = 0,\ \Phi_2(x_2) = 0 \tag{29}$$

We do not yet know whether Φ_1 and Φ_2 are linearly independent.

Comparing (29) with (28) we see that g is of the form

$$g(x,\xi) = H(\xi - x).C.\Phi_1(x) + H(x-\xi).D.\Phi_2(x) \tag{30}$$

since this satisfies both the BC's, whatever the values of the constant C and D.

C and D are determined by matching across $x = \xi$, exactly as for the IVP in Section (2). The continuity and jump conditions are identically the same as before; they yield, respectively,

$$D\Phi_2(\xi) - C\Phi_1(\xi) = 0$$

and

$$D\Phi_2'(\xi) - C\Phi_1'(\xi) = -1$$

Thus

$$\begin{bmatrix} -\Phi_1(\xi) & \Phi_2(\xi) \\ -\Phi_1'(\xi) & \Phi_2'(\xi) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{31}$$

Naturally one hopes that (31) has a unique solution. The condition for this is

$$\begin{aligned} \det[\ ] &= \{-\Phi_1(\xi)\Phi_2'(\xi) + \Phi_2(\xi)\Phi_1'(\xi)\} \\ &= -w\{\Phi_1(\xi), \Phi_2(\xi)\} \equiv -w(\xi) \neq 0 \end{aligned}$$

If it is satisfied, then the solution reads

$$D = -\frac{\Phi_1(\xi)}{w(\xi)},\ C = -\frac{\Phi_2(\xi)}{w(\xi)}$$

Whence

$$g(x,\xi) = -\frac{\{H(\xi - x)\Phi_2(\xi)\Phi_1(x) + H(x-\xi)\Phi_1(\xi)\Phi_2(x)\}}{\{\Phi_1(\xi)\Phi_2'(\xi) - \Phi_1'(\xi)\Phi_2(\xi)\}} \tag{32}$$

In the general case, $W \neq 0$, our problem is solved, by (4,10). In full, the solution reads

$$\Psi(x) = -\int_{x_1}^{x_2} f(\xi) \frac{\left\{ \begin{array}{l} H(\xi - x)\Phi_2(\xi)\Phi_1(x) \\ +H(x-\xi)\Phi_1(\xi)\Phi_2(x) \end{array} \right\}}{w(\xi)} d\xi \tag{33}$$

In the special case, $W = 0$ expresses the fact that Φ_1 and Φ_2 defined by (26) are actually the same function: in other words, the corresponding homogeneous problem ($L\Phi = 0, \Phi(x_1) = 0 = \Phi(x_2)$) then has a solution (satisfying both BC's and not only one). In this special case, g does not exist, whence our attempt at a solution in the form (4) fails. Again there are two possibilities:

(i) Either the general special case, where the inhomogeneous problem has no solution; in fact it has none unless $f(x)$ satisfies some constraints;
(ii) Or the special case, where $f(x)$ does satisfy these constraints.

Then our inhomogeneous problem has solution, but they are not unique: in fact there are infinitely many. Obviously so: for if we have one solution to the inhomogeneous problem, then we can add to it any numerical multiple of the solution to the corresponding homogeneous problem and the result still satisfies all the requirements (i.e. both the equation and the BC's).

NUMERICAL METHOD OF GREEN'S FUNCTION

There have been several methods for numerical solution of boundary value problems with ordinary differential equation [18]. The shooting method is a traditional one and the implicit finite difference method is also one of the useful tools [15-18, 21-23].

In [22, 23] proposed since collocation methods based on the double exponential transformation.

A method for numerical solution of boundary value problems based on the classical method of Green's function incorporated with the double exponential transformation is presented. Although the method of Green's function is a classical one for analytical manipulation of solution of boundary value problems with differential equation, the method presented here gives an approximate solution of very high accuracy with a small number of function evaluations.

The differential equation we consider here is

$$L[u]+f(x)=0,\ a<x<b \tag{34}$$

where u(x) is the function to be determined and L[u] is a self-adjoint operator associated with the Sturm-Liouville eigen-value problem defined as

$$L[u]\equiv\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\}+q(x)u \tag{35}$$

We assume here that p(x), q(x) and *f*(x) are known and analytic on (a,b) and *f*(x) may be singular at x = a,b provided that the equation is defined properly. We also assume that p(x)>0 on (a,b). For the moment we assume that the boundary condition is homogeneous, i.e.

$$u(a)=0,\ u(b)=0$$

However, if p(x) vanishes at x = a or x = b we can impose there an inhomogeneous boundary condition

$$u(a)=\text{finite}\neq 0,\ u(b)=\text{finite}\neq 0$$

The integral we want to evaluate is (11), i.e.

$$\begin{aligned}u(x)&=\int_a^x f(\xi)g(x,\xi)d\xi+\int_x^b f(\xi)g(x,\xi)d\xi\\&=J_1(x)+J_2(x)\end{aligned} \tag{36}$$

from (10).

If we approximate $J_1(x)$ and $J_2(x)$ by double exponential formula, abbreviated as the DE formula, for indefinite integral [18, 24], we immediately have (mesh size h)

$$\begin{aligned}u(x)=\ &y_2(x)h\sum_{j=-N}^{N}y_1(\psi(jh))f(\psi(jh))\psi'(jh).\\&\left(\frac{1}{2}+\frac{1}{\pi}\mathrm{Si}\left(\pi\frac{\psi^{-1}(x)}{h}-xj\right)\right)\\&+y_1(x)h\sum_{j=-N}^{N}y_2(\psi(jh))f(\psi(jh))\psi'(jh)\\&\left(\frac{1}{2}-\frac{1}{\pi}\mathrm{Si}\left(\pi\frac{\psi^{-1}(x)}{h}-xj\right)\right)+E_N\end{aligned} \tag{37}$$

Where

$$\xi=\psi(t)=\frac{b-a}{2}\tanh\left(\frac{\pi}{2}.\sinh t\right)+\frac{b+a}{2}$$

and d>0 is interval when, $g(\psi(t))\psi'(t)$ is regular in the strip |Im t|<d and

$$\mathrm{Si}(t)=\int_0^t\frac{\sin\tau}{\tau}d\tau$$

and

$$N=\frac{1}{h}\log\frac{2d}{h(\alpha-\varepsilon)} \tag{38}$$

and E_N is the error term given by

$$E_N=O\left(\exp\left(-\frac{\pi N d}{\log(2Nd/(\alpha-\varepsilon))}\right)\right) \tag{39}$$

We first gave h and then determined N by (38). However, if we want to give first N then to determine h, it should be determined

$$h=\frac{1}{N}\log\frac{2Nd}{\alpha-\varepsilon} \tag{40}$$

This is the DE formula for numerical solution of the boundary value problem (9) with homogeneous boundary condition. Since we can compute $y_1(\psi(jh))f(\psi(jh))\psi'(jh)$ and $y_2(\psi(jh))f(\psi(jh))\psi'(jh)$ beforehand for each j, what we have only to compute for a given x is $y_1(x), y_2(x)$ and $1/2+\mathrm{Si}(\pi\psi^{-1}(x)/h-xj)/\pi$ and their product sum. Thus, this formula for x consists of evaluations of simple functions and their sum, so that the present method is suitable for parallel computation. Also note that if x is equal to one of the sinc points, i.e., $x=\psi(kh)$ for some integer k,

$$\mathrm{Si}(\pi\psi^{-1}(x)/h-xj)=\mathrm{Si}(\pi(k-j))$$

holds and computation of Si becomes very simple.

RESULTS AND DISCUSSION

Infinite domain: Consider the case when the source term is zero and the volume of interest is the infinite domain, so that the surface integral is zero. Then we have

$$u(\vec{r}_0,t)=a\int_v u_0(\vec{r})G(\vec{r}\,|\,\vec{r}_0,t)d^3\vec{r}$$

In one dimensional, this reduces to

$$u(x_0, t) = \sqrt{\frac{a}{4\pi t}} \int_{-\infty}^{\infty} \exp\left(-a \frac{(x - x_0)^2}{4t} \right) u_0(x)\, dx \quad t > 0$$

Thus we see that the field u at a time t>0 is given by the convolution of the field at time t = 0 with the Gaussian function

$$\sqrt{\frac{a}{4\pi t}} \exp\left(-\frac{ax^2}{4t} \right)$$

Numerical examples: The interval of definition of the problem is (0, 1). We first chose N = 4,8,16,32,... and computed h by (40). For each N we evaluated u(x) by means of the formula (37) for

$$x = 0.01, 0.02, 0.03, \ldots, 0.97, 0.98, 0.99 \qquad (41)$$

and picked up the maximum absolute value of the error. Since in the integrand of (37) the singular points which lie nearest to the real axis are the poles of

$$\psi'(t) = \frac{\pi}{4} \frac{\cosh t}{\cosh^2(\pi/2 \sinh t)}$$

with the distance $\pi/2$, we set $d = \pi/2$. In order to show the high efficiency of the present formula we carried out numerical computation with quadruple precision arithmetic.

Let

$$\begin{cases} \dfrac{d^2u}{dx^2} - \dfrac{3}{4}\left(x^{-\frac{1}{2}} + (1-x)^{-\frac{1}{2}} \right) = 0 \\ u(0) = 1 \quad , \quad u(1) = 0 \end{cases} \qquad (42)$$

Here we take

$$L[u] = \frac{d^2u}{dx^2} \qquad (43)$$

and the Green's function corresponding to (43) is

$$y_1(x) = x,\; y_2(x) = 1 - x \qquad (44)$$

i.e.

$$g(x,\xi) = \begin{cases} \xi(1-x) & \xi \le x \\ x(1-\xi) & x \le \xi \end{cases} \qquad (45)$$

The exact solution of this problem is

$$u(x) = x^{\frac{3}{2}} + (1-x)^{\frac{3}{2}} - 1 \qquad (46)$$

In this example, as ξ tends to 0,

$$f(\xi) = -3/4\left(\xi^{-1/2} + (1-\xi)^{-1/2} \right) = O\left(\xi^{-1/2} \right)$$

from (42) and $y_1(\xi) = O(\xi)$ from (44), so that

$$g(\xi) = y_2(\xi) f(\xi)\, y_1(\xi) = O(\xi^{1/2}) = O(\xi^{-1+\alpha}), \alpha = 3/2$$

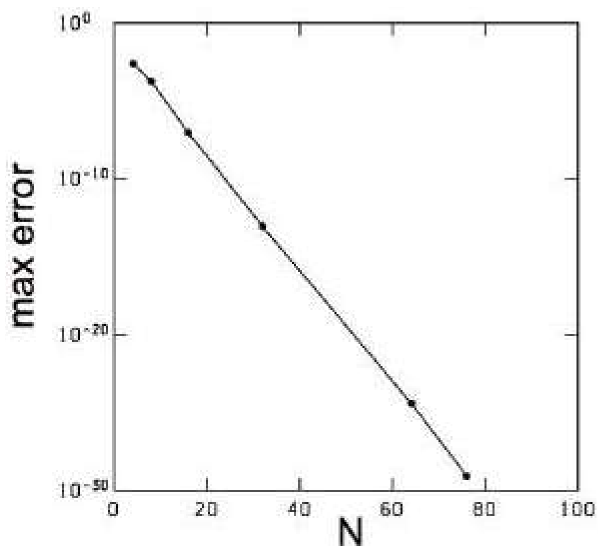

Fig. 2: The maximum error

holds. In a similar way we see that as ξ tends to 1 $g(\xi) = O((1-\xi)^{1/2})$. In this way we chose α = 3/2 for (38), (39) and (40).

We chose N = 4,8,16,32,64,76 computed numerical solution for x's given in (41) and plotted the maximum absolute error of the numerical solution as a function of N in Fig. 2. Although $f(x)$ of this problem is singular at x = 0 and 1, the result is very good. Actually, the error decays almost exponentially as given in (39) and attains about 10^{-30} with N = 76.

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