Middle-East Journal of Scientific Research 11 (1): 128-133, 2012
ISSN 1990-9233

Automatic Continuity of Multiplicative Linear Functional On LMC Spaces

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Abstract: Functional, Linear Functional, Multiplicative linear Functional and research on continuity of these Functional are very important in advanced studies on topological algebras and mathematical analysis. Our main purpose will be researching to obtain some results about automatic continuity and boundedness of these functional on LMC (locally multiplicatively convex) spaces. We will try on Ernest.A.Michael's questions. He introduced two problems in 1952 which nobody has been found the exact answers yet.
These questions are:

1. Is every commutative F-Algebra functionally continuous?
2. Is every multiplicative linear functional on a commutative complete locally m-convex algebra bounded?

Key word: Complete locally m-convex . Ernest. A. Michael's questions

INTRODUCTION

Some main definitions and concepts we must know are outlined in first step (Section2). In the next step we will recognize some main properties of these functionals. Especially we will answer the question "What are the necessary conditions for a topological algebra X to linear functionals on X be multiplicative?"[1-5] But can we always define any homomorphism on a topological space? What are the main conditions for these spaces to accept these type of functionals? Results are in section 4. And in the final step the automatic continuity conditions of these functionals are under test and investigation [6-9].

SEVERAL BASIC DEFINITIONS

Definition 2.1: On a space X with field such $\mathcal{F}$ (usually $\mathbb{C}$) a linear operator $\phi:X\to\mathcal{F}$ called multiplicative if $\phi(xy)=\phi(x)\phi(y)\ \forall\ x,y\in X$.

Definition 2.2: Let $(X,\mathcal{T})$ is a topological vector space. Then

a) X is locally convex if it has a locally base with convex elements.
b) X is locally bounded if there is a bounded neighborhood of 0 in X.

Definition 2.3: $(X,\mathcal{T})$ is a F-Space when it is a complete metrizable topological vector space.

Definition 2.4: X is a Frechet space when X be a locally convex F-Space.

Definition 2.5: A subset U from space(algebra) A called idempotent if $UU\subset U$.

Definition 2.6: A subset U from space(algebra) A called m-convex (multiplicatively convex) when U is convex and idempotent.

Definition 2.7: A topological algebra A called locally multiplicatively convex if there is a base of neighborhood of 0 which contains symmetric m-convex subsets. We show this A with LMC Algebra.

WHEN AND WHERE A LINEAR FUNCTIONAL IS MULTIPLICATIVE?

$$\exp(a)=\sum_{n=0}^{\infty}\frac{a^n}{n!}$$

$$\exp(A)=\{\exp(x): x\in A\}$$

$\phi_A=\{\phi:A\to\mathcal{F}\mid \phi \text{ is multiplicative}\}$ (A is an algebra on field $\mathcal{F}$).
We know for every $\phi\in\phi_A$:

1. $\phi(1_A)=1$
2. $\phi(a)\neq 0\ (\forall a\in A^{-1})$
3. $\forall x\in A, \forall\phi\in\phi_A; \phi(\exp(x))\neq 0$

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Theorem 3.1: Let A be a commutative Banach algebra with 1 and ϕ a Linear Functional on A such that

$$\forall x \in A, \phi(\exp(x)) \neq 0, \phi(1) = 1$$

Then $\phi \in \phi_A$ that means ϕ is multiplicative.

Proof: At first we show that ϕ is continues on A. Let $x \in A$ with $\|x\|<1$. (Lemma: $x \in A$, $\|1-x\|<1 \Rightarrow (\exists y \in A, \exp(y) = x)$) so If $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$ then

$$\left\|1-(1-\frac{x}{\lambda})\right\| = \left\|\frac{x}{\lambda}\right\| = \frac{\|x\|}{|\lambda|} < 1$$

and so

$$1-\frac{x}{\lambda} \in \exp(A)$$

and

$$\phi(1-\frac{x}{\lambda}) \neq 0$$

this means

$$\lambda - \phi(x) \neq 0 \ (\forall |\lambda| \geq 1)$$

But $\|\phi\|=1$ so $|\phi(x)|<!$. In fact

$$\forall x \in A \quad \|x\| < 1 \Rightarrow |\phi(x)| < 1$$

Now if $0<\varepsilon$ we take $\delta=\varepsilon$ and then

$$\forall x \ (\|x\| < \delta = \varepsilon \Rightarrow \|\varepsilon^{-1}x\| < 1 \Rightarrow |\phi(\varepsilon^{-1}x)| < 1 \Rightarrow \phi|(x)| < \varepsilon)$$

that means ϕ is continues on A. Now we show that

$$\forall x \in A, \phi(x^2) = (\phi(x))^2$$

Let $x \in A$ and $\forall \lambda \in \mathbb{C}$,

$$F(\lambda) = \phi(\exp(\lambda x))$$

So $F: \mathbb{C} \to \mathbb{C}$ and

$$F(\lambda) = \phi(\sum_{\circ}^{\infty} \frac{\lambda^n x^n}{n!}) = \sum_{\circ}^{\infty} \frac{\phi(\lambda^n x^n)}{n!}$$

so F is an Entire Function such that $\forall \lambda \in \mathbb{C}$, $f(\lambda) \neq 0$. Also

$$|F(\lambda)| \leq (\sum_{\circ}^{\infty} \frac{\|x\|^n |\lambda|^n}{n!}) = \exp(|\lambda| \ \|x\|)$$

so we can write F = exp(G) in which G is an Entire Function such that

$$\forall \lambda \in \mathbb{C}; \operatorname{Re} G(\lambda) \leq \|x\| \ |\lambda|$$

and since F(0)=1 from real part of Lyovil Theorem we can say G(0)=0. So for a constant value such t we can write G = tz and

$$F(z) = \exp(G(Z)) = e^{tz} = \sum_{\circ}^{\infty} \frac{t^n}{n!} z^n \Rightarrow F(z) = \sum_{\circ}^{\infty} \frac{t^n}{n!} z^n$$

The result is

$$(\forall n \in N): \phi(x^n) = t^n = (\phi(x))^n$$

and for n=2 we have $\phi(x^2) = (\phi(x))^2$. Now ϕ is multiplicative because:

$$\begin{aligned}
&\forall x, y \in A: \\
&\phi(xy) = (\phi(\frac{1}{4}[(x+y)^2 - (x-y)^2]) \\
&= \frac{1}{4}(\phi[(x+y)^2)) - \phi(x-y)^2) = \frac{1}{4}(\phi((x+y)^2 - (\phi(x-y)^2) \\
&= \frac{1}{4}(\phi(x) + \phi(y)^2 - (\phi(x) - \phi(y)^2) \\
&= \frac{1}{4}(\phi(x)^2 + (\phi(y)^2) + 2(\phi(x)\phi(y)) \\
&\quad - (\phi(x)^2) - (\phi(y)^2) + 2\phi(x)\phi(y)) \\
&= \frac{1}{4}(4\phi(x)\phi(y)) = \phi(x)\phi(y)
\end{aligned}$$

Theorem 3.2: (Gleason-Kahan-Zelazko) Let A be a complex Banach algebra with unit element e and ϕ be a linear functional on A such that $\phi(e) = 1$. If $a \in A^{-1}$, $\phi(a) \neq 0$ Then $\forall x,y \in A$: $\phi(xy) = \phi(x)\phi(y)$.

Definition 3.3: An commutative Banach algebra B is Semi-simple when

$$\forall b \in B \setminus \{0\}, \exists \phi \in \phi_B, \phi(b) \neq 0$$

Theorem 3.4: Let A is a complex Banach algebra and B is an commutative semi-simple Banach algebra and T: $A \to B$ be a linear function such that $T(e_A) = e_B$ and $\forall a \in A^{-1}; T(a) \in B^{-1}$ Then T is Multiplicative.

Proof: Let $G \in \phi_B$, a_1, $a_2 \in A$ so GOT is a linear functional such that

$$\forall a \in A^{-1}, GOT(a) = G(T(a)) = G(b)$$

where $b \in B^{-1}$. so GOT(a)≠0 because $a \in A^{-1}$ and GOT is linear functional and so GOT is multiplicative from the

Gleason-Kahan-Zelazko theorem. Therefore

$$G(T(a_1a_2) - T(a_1)T(a_2)) = GOT(a_1)GOT(a_2)$$

so

$$T(a_1a_2) - T(a_1)T(a_2) = 0$$

because B is semi-simple. And this means T is Multiplicative.

IS THERE EXIST A MULTIPLICATIVE LINEAR FUNCTIONAL ON EVERY TOPOLOGICAL ALGEBRA?

We can always define a multiplicative linear functional on a commutative complete normed algebra(commutative Banach algebra). In fact:

Theorem 4.1: Let I be a maximal modular ideal with co-dimension 1 for a Banach algebra A Then there is a multiplicative linear functional ϕ:A$\rightarrow\mathcal{F}$ such that I=ker(ϕ).

But now we find some examples will show us these functionals cannot always be defined on non-commutative spaces:

Example 4.2: Let

$$A = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}; a, b, c \in \mathbb{C}\}$$

Clearly A is a non-commutative algebra with unit element $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and dim (A) = 3 and with a norm such as

$$\left\| \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right\| = \sqrt{a^2 + b^2 + c^2} \qquad \forall \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in A$$

A is a Banach algebra. We can define $\phi_1 : A \rightarrow \mathbb{C}$ with $\phi_1(M) = m_1$ when

$$\begin{bmatrix} m_1 & m_2 \\ 0 & m_3 \end{bmatrix} = M \in A$$

It is easy to show that ϕ_1 is a multiplicative linear functional.

Example 4.3: Let

$$A = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in C\}$$

A is a non-commutative algebra with unit element $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But you never can define a multiplicative linear functional on A !. In fact if A be an algebra of square matrices of order n, then there is no multiplicative linear functional ϕ:A$\rightarrow\mathcal{F}$.

Another spaces always accept these functionals are LMC spaces. In the other hand:

Theorem 4.4: Let A be a commutative LMC algebra and I a closed regular maximal ideal of A. Then there is a continues multiplicative linear functional ϕ:A$\rightarrow\mathcal{F}$ such that ker(ϕ)=I.

Proof: Every LMC algebra such A is a normed division space so A and complex numbers algebra($\mathbb{C}$) are isomorphic. So there is no Homeomorphism on A unless the Identity function ϕ (x)=x ($\forall$x$\in$A) and clearly ϕ is a continues multiplicative linear functional. Now theorem4.1 gives us ker(ϕ)=I. (Remark: Every Homeomorphism is a continues multiplicative linear functional.)

AUTOMATIC CONTINUITY OF MULTIPLICATIVE LINEAR FUNCTIONALS

Definition 5.1: A topological space X is F.C. (Functionally continues), when every multiplicative linear functional on X be continues.

At first step we show an arbitrary topological space is not F.C. necessary.

Example 5.2: Let A=C([0,1]) algebra of all continues functions on [0,1] and

$$P_t(f) = |f(t)|, (0 \langle t \leq 1)$$

Every P_t:A$\rightarrow\mathbb{R}^+$ is a Semi-norm and A is a topological algebra with the topology generated by the family of {P_t: 0<t≤1}. Define ϕ:A$\rightarrow$ $\mathbb{C}$ with ϕ (f)=f(o). ϕ is a multiplicative linear functional on A that is not continues.

Proof: $\forall f, g \in A, \forall \alpha \in \mathbb{C}$: ϕ (αf+g)= (αf+g)(0)=αf(0)+ g(0)= α ϕ (f)+ϕ (g) and ϕ (fg)=(f.g)(0)=f(0).g(0)= ϕ (f). ϕ (g). Let

$$V(p_i, n) = \{f \in A : p_i(f) < \frac{1}{n}\}$$

and

$$S = \{V(P, n) : n \in N, t \in N, t \in (0,1]\}$$

S is a base for the topology defined on A. Now we show if $f_0 \epsilon A$ then ϕ is not continues at f_0. It is enough to prove

$$\exists \varepsilon > 0, \forall t, \forall n, \exists f \in A; p_t(f - f_0) < \frac{1}{n} \ \& \ |\phi(f - f_0)| \geq \varepsilon$$

And cause of A is a topological space that is enough to prove ϕ is not continues at 0. That means

$$\exists \epsilon > 0, \forall t \in (0,1], \forall n \in N, \exists f \in A$$

$$p_t(f) < \frac{1}{n} \ \& \ |\phi(f)| \geq \varepsilon$$

Let $\varepsilon = \frac{1}{2}$, $0<t\leq 1$ and $n \in N$. So

$$\exists K \in N; (1+t)^{-k} < \frac{1}{n}$$

now let $f:[0,1] \to \mathbb{C}$ with

$$f(x) = \frac{1}{(1+x)^k}$$

$f \in A$ and

$$p_t(f) = |f(t)| = \left|(t+1)^{-k} < \frac{1}{n}\right|$$

then we can write

$$|\phi(f)| = |f(0)| = 1 \geq \frac{1}{2} = \varepsilon$$

that means ϕ is not continues at 0 and so on A.

Remark 5.3

a) Conditions Continuity and Boundedness of operators on Normed spaces are equivalent.
b) Continuity on a Normed space such X and Continuity at a point of X are equivalent.

Therefore even normed topological algebras are not F.C. necessary. In fact it seems completeness is a necessary condition for normed spaces to be F.C.

Example 5.4: Let A={P(x): $x \in [0,1]$, P is a polynomial}. Define the supermom-norm

$$\| \ \| : A \to R$$
$$\|P\| = SUP\{|P(x) : x \in [0,1]|\}$$

So A is a normed algebra with this norm. We will show A is not complete and therefore is not F.C.

Proof: Let

$$\forall n \in N, P_n(x) = \sum_{k=1}^{n} \frac{x^k}{k!}$$

($x \in [0,1]$) so $\forall n$, $P_n \epsilon A$ and (P_n) is a Cauchy sequence convergence to e^x. In fact $P_n(x) \to e^x$ but $e^x \notin A$ that shows A is not complete. Now let $\phi : A \to \mathbb{C}$ with $\phi(p) = P(a)$. Clearly ϕ is multiplicative linear functional but we will prove that ϕ is not continues. From Remark5.3. it is enough to show ϕ is not bounded. We have

$$\|\phi\| = \sup\{|\phi(p)| : P \in A, \|P\| \leq 1\}$$

so with definition $p_n(x) = e^n$, (P_n) is a sequence with elements of A such that $\|p_n\| \leq 1$ and

$$|\phi(p_n)| = |2^n| \to \infty$$

so ϕ is not bounded. Equivalently ϕ is not continues and so A is not F.C.

It has been proved that every commutative Banach algebra is F.C.
In particular

Theorem 5.5: Let A be a commutative Banach algebra and ϕ a multiplicative linear functional on A. Then:

a) ϕ is continues and $\|\phi\| \leq 1$.
b) If A be unital then $\phi(1_A) = 1 = \|1_A\|$.

In the extended position we claim following theorem:

Theorem 5.6: If ϕ:B→A be a multiplicative linear functional when B is a commutative Banach algebra and A is a semi-simple commutative Banach algebra, then ϕ is automatically continues.

Proof: Let (x_n) be a sequence in B such that $x_n \to x$. Cause of completeness of B, $x \in B$. Follow the sequence $\phi(x_n)$ in A and suppose $\phi(x_n) \to y$. From the closed graph theorem it is enough to prove $y = \phi(x)$. Let $h \in \phi_A$ and $\psi = h o \phi$ therefore $\psi \in \phi_B$ because:

$$\begin{aligned} \forall t, \forall b \in B, \psi(tb) &= ho\phi(tb) = h(\phi(tb)) = h(\phi(t).\phi(b)) \\ &= (ho\phi(t)).(ho\phi(b)) = \psi(t).\psi(b) \end{aligned}$$

and

$$\begin{aligned} \psi(\alpha t + b) &= (ho\phi)(\alpha t + b) = h(\alpha\phi(t) + \phi(b)) \\ &= \alpha(ho\phi(t) + ho\phi(b)) = \alpha\psi(t) + \psi(b) \end{aligned}$$

A and B is Banach algebras so h and ψ are continues and so

$$
\begin{aligned}
h(y) &= h(\lim\phi(x_n)) = \lim\psi(x_n) = \psi(\lim x_n) \\
&= \psi(x) = h(\phi(x))\ (\forall h \in \phi_A) \\
&\Rightarrow h(y-\phi(x)) = 0 (\forall h \in \phi_A) \Rightarrow y-\phi(x) \in \mathrm{rad}(A)
\end{aligned}
$$

A is semi-simple so rad (A) = {0} and then y-ϕ(x) = 0 that means y = ϕ(x).

Following example shows that an LMC space is not necessary F.C.

Example 5.7: Let T be a non-compact set but locally compact or prime countable set and A=C(T) with compact-open topology. Then

a) A is a commutative complete LMC algebra.
b) A is not F.C.
c) Every multiplicative linear functional on A is Bounded.

Theorem 5.8: Let A be a LMC algebra and

$$R(A) = \{x \in A : f(x) = 0\ ,\forall f \in \phi_A\}$$

1) If A be complete and commutative and algebra A/R(A) be F.C. then A is F.C.
2) If A be an ideal of LMC algebra B and B be F.C. then A is F.C.

Proof

1) Let f be a multiplicative linear functional on A. so $\forall x \in R(A), f(x) = 0$. so if π:A→ A/R(A) be the natural mapping (π is multiplicative and always continues) then there is a multiplicative linear functional g on A/R(A) such that f = goπ. Cause of A/R(A) is F.C. then g is continues and so f = goπ is continues. Therefore A is F.C.
2) Let f be a multiplicative linear functional on A. The algebra $\mathbb{C}$ is unital and $f : A \to \mathbb{C}$ is a Homomorphism. So there is a an Extension of f such g:B→$\mathbb{C}$ that g is multiplicative linear functional too. B is F.C. so g is continues on B. Therefore $f \subseteq g$ is continues too. Then A is F.C.

Definition 5.9: A topological vector space X is Fundamental, when there is a b>1 such that for every sequence (x_n) in X, $b^n(x_n-x_{n-1})$→0 if and only if (x_n) be a Cauchy sequence.

Definition 5.10: A Fundamental topological algebra X is called FLM (Fundamental locally multiplicative)

$$\exists U_0 \in N(\circ), \forall V \in N(\circ),\ \exists m \in N, \forall K \geq m,\ U_0{}^k \subset V$$

Theorem 5.11: Let A be a complete metrizable FLM algebra and ϕ:A→¢ a multiplicative linear functional on A. then ϕ is automatically continues. In the other hand every complete metrizable FLM algebra is F.C.

Proof: Let ϕ be a nonzero multiplicative linear functional on A, b>1 and x∈A such that $b^n x^n$→0. Let

$$S_n = \sum_{k=1}^{n} x^k$$

A is complete and (S_n) a convergence sequence so there is a

$$y = \sum_{k=1}^{n} x^k \in A$$

and we can write (note that product operator is continues):

$$
\begin{aligned}
y - xy &= \lim S_n - x\lim S_n = \lim(S_n - x_{S_n}) \\
&= \lim(\sum_{k=1}^{n} x^k - x\sum_{k=1}^{n} x^k) = \lim(\sum_{k=1}^{n} x^k - x\sum_{k=2}^{n+1} x^k) \\
&= x - \lim x^{n+1} = x
\end{aligned}
$$

Since ϕ is linear and multiplicative so ϕ (x)≠1 because if ϕ (x)=1 then 1=ϕ (x) = ϕ (y-yx) = ϕ (y)- ϕ (x) ϕ (y) = 0 that is impossible. So $b^n x^n$→0 ⇒ϕ (x)≠1. Now if |ϕ(x)|>1 with choose $X_\circ = \frac{x}{\phi(x)}$ we have

$$b^n x_\circ{}^n = b^n \frac{x^n}{(\phi(x))^n} = \frac{1}{(\phi(x))^n} b^n x^n \to \circ$$

so from above $\phi(x_o)$≠1 but

$$\phi(x_\circ) = \frac{1}{\phi(x)}\phi(x) = 1$$

that isn't possible. So it has been proved that:
$b^n x^n$→0 ⇒ |ϕ(x)|<1. now if (x_n) be a null sequence(x_n→0) in A and x>0, since

$$(\exists K_\circ \in N, \forall K \geq \underset{\circ}{K}; U_\circ^K \subset V)$$

there is n∈Z^+ such that $b\varepsilon^{-1}x_n \in U_0$. so if V be a neighborhood of 0 then

$$(\exists K_\circ \in N, \forall K \geq \underset{\circ}{K}; U_\circ^K \subset V)$$

and then $b^k(\varepsilon^{-k}x_n^k) \in V$ but V was arbitrary that means

$$\lim b^k(\varepsilon^{-1} x_n)^k \to$$

so $|\phi(\varepsilon^{-1} x_n)|<1$ that means $|\phi(x_n)|<\varepsilon$ equivalently $\phi(x_n)\to 0$. in fact we proved : $x_n\to 0 \Rightarrow \phi(x_n)\to 0$ that means ϕ is continues [10, 11].

CONCLUSION

Clearly a complete metrizable FLM algebra is an F-Algebra (Frechet Algebra). So multiplicative linear functional are automatically continues on this subset of F-Algebras. In fact for most approaching to the answers of michael's questions we can say:

Conclusion 6.1: Every F-Algebra with property

$$\exists U_\circ \in N(\circ), \forall V \in N(\circ), \exists K_\circ \in N, \forall K \geq K_\circ : U_\circ^k \subseteq V$$

is functionally continues.

Furthermore it is easy to observe this fact that every locally bounded topological algebra is a FLM-Algebra. So from above concepts we can also say:

Conclusion 6.2: Every complete locally bounded topological algebra is a functionally continues F-Algebra.

At the other hand every locally bounded F-Algebra is F.C. Therefore multiplicative linear functional are automatically continues on this big subset of F-Algebras.

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