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The General Solutions of m×n Fuzzy Linear Systems

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Abstract: The m×n fuzzy linear systems are studied in [1-3] based on Friedman's method. Recently, Ezzati [3] proposed a new method for solving fuzzy linear systems, numerically better than Friedman's method. The main aim of this paper, is to develop Ezzati's method for solving m×n fuzzy linear systems. Numerical examples are used to illustrate the proposed model.

Key words: The m×n fuzzy linear system . general solutions . embedding method

INTRODUCTION

Consider the fuzzy linear system

$$A\tilde{X} = \tilde{Y} \qquad (1)$$

where $A \in R^{m\times n}$ and $\tilde{X}, \tilde{Y} \in \mathbf{E}^1$ are fuzzy number vectors. Friedman *et al*. [4] proposed a general model for solving fuzzy linear systems by using the embedding approach. They replace the original m×n fuzzy linear system by 2n×2n crisp linear system. Based on their work, Allahviranloo *et al*. [1], Asady *et al*. [2] and Bing Zheng *et al*. [3] have developed a method for solving m×n fuzzy linear systems for m≤n. Also, fuzzy matrix equation in the form AXB = C was introduced by Allahviranloo *et al*. [5].

Recently, Ezzati [6] proposed a new method for solving fuzzy linear systems. He replace the original n×n fuzzy linear system by two n×n crisp linear systems. It is clear that, solving two n×n linear system is better than solving 2n×2n linear system and hence, Ezzati's method is numerically better than Friedman's method.

Based on their work, in this paper, we give a novel method for solving a m×n fuzzy linear system by using the embedding method and replace the original m×n fuzzy linear system by two m×n crisp and consistent linear systems. Our method is numerically better than Friedman's method.

This paper is organized as follows:

In section 2, we study the Friedman's method and Ezzati's method for solving the fuzzy linear systems. The proposed model for solving m×n fuzzy linear systems and its properties are discussed in section 3. In section 4, the proposed model is illustrated by solving some numerical examples. The conclusion is drawn in section 5.

PRELIMINARIES

Fuzzy numbers and their operations: Definition 1 [4, 7] An arbitrary fuzzy number $\tilde{u}$ is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$; $0 \le r \le 1$ which satisfy the following requirements [4, 7]:

- $\underline{u}(r)$ is a bounded monotonic increasing left continuous function;
- $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function;
- $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

The set of all these fuzzy numbers is denoted by E^1. A crisp number k is simply represented by $\overline{u}(r) = \underline{u}(r) = k; 0 \le r \le 1$ and called singleton.

For arbitrary $\tilde{u} = (\underline{u}(r), \overline{u}(r))$, $\tilde{v} = (\underline{v}(r), \overline{v}(r))$ and scalar k we define addition $(\tilde{u} + \tilde{v})$, subtraction and scalar multiplication by k as

Addition:

$$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r)$$

subtraction:

$$(\underline{u-v})(r) = \underline{u}(r) - \overline{v}(r), (\overline{u-v})(r) = \overline{u}(r) - \underline{v}(r)$$

Scalar multiplication:

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$$k\tilde{u}=\begin{cases}(k\underline{u}(r),k\overline{u}(r)), & k\geq 0\\ (k\overline{u}(r),k\underline{u}(r)), & k<0\end{cases}$$

For two arbitrary fuzzy numbers $\tilde{x}=(\underline{x}(r),\overline{x}(r))$ and $\tilde{y}=(\underline{y}(r),\overline{y}(r))$, $\tilde{x}=\tilde{y}$ if and only if $\underline{x}(r)=\underline{y}(r)$ and $\overline{x}(r)=\overline{y}(r)$.

Definition 2: The m×n linear system of equations

$$\begin{cases}a_{11}\tilde{x}_1+\cdots+a_{1n}\tilde{x}_n=\tilde{b}_1,\\ a_{21}\tilde{x}_1+\cdots+a_{2n}\tilde{x}_n=\tilde{b}_2,\\ \quad\vdots\\ a_{m1}\tilde{x}_1+\cdots+a_{mn}\tilde{x}_n=\tilde{b}_m,\end{cases}\qquad (2)$$

where the coefficients matrix $A=[a_{ij}],\ 1\leq i\leq m, 1\leq j\leq n$ is a crisp m×n matrix and $\tilde{y}_i\in E^1,\ 1\leq i\leq m$ and the unknowns $\tilde{x}_j\in E^1,\ 1\leq j\leq n$ is called a general fuzzy linear system (GFLS).

Definition 3: [8] A fuzzy number vector $(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n)^T$ given by $\tilde{x}_j=(\underline{x}_j(r),\overline{x}_j(r)),\ 1\leq j\leq n,\ 0\leq r\leq 1$, is called a solution of (2) if

$$\underline{\sum_{j=1}^{n}a_{ij}x_j}(r)=\sum_{j=1}^{n}\underline{a_{ij}x_j}(r)=\underline{b}_i(r),$$

$$i=1,\cdots,m \qquad (3)$$

$$\overline{\sum_{j=1}^{n}a_{ij}x_j}(r)=\sum_{j=1}^{n}\overline{a_{ij}x_j}(r)=\overline{b}_i(r),$$

and if, for a particular i, $a_{ij}>0$, $1\leq j\leq n$ we simply get

$$\sum_{j=1}^{n}a_{ij}\underline{x}_j=\underline{y}_i,\quad \sum_{j=1}^{n}a_{ij}\overline{x}_j=\overline{y}_i$$

Finally, we conclude this section by a reviewing on the proposed methods for solving fuzzy linear system in Friedman *et al*. [4] and Ezzati [6].

Friedman's method: Friedman *et al*. [4] wrote the Eq. (3) as:

$$S_{2m\times 2n}X=Y \qquad (4)$$

where the elements of $S=(s_{ij})$, $1\leq i\leq 2m$, $1\leq j\leq 2n$, were defined by:

$$\text{if } a_{ij}\geq 0,\ s_{ij}=s_{i+m,j+n}=a_{ij}\quad \text{if } a_{ij}<0,\ s_{i,j+n}=s_{i+m,j}=-a_{ij} \qquad (5)$$

and any s_{ij} is not determined by Eq. (5) is zero, the unknowns and the right-hand side column were

$$X=(\underline{x}_1,\underline{x}_2,\cdots,\underline{x}_{2n},-\overline{x}_1,-\overline{x}_2,\cdots,-\overline{x}_{2n})^T$$

$$Y=(\underline{y}_1,\underline{y}_2,\cdots,\underline{y}_{2m},-\overline{y}_1,-\overline{y}_2,\cdots,-\overline{y}_{2m})^T$$

respectively. The structure of S implies that $s_{ij}\geq 0, 1\leq i\leq 2m$, $1\leq i\leq 2n$ and that

$$S=\begin{pmatrix}B & C\\ C & B\end{pmatrix}$$

where B contains the positive entries of A and C the absolute values of the negative entries of A and A = B-C.

We know that $X=S^{-}Y$ is an arbitrary solution for the Eq(4) but this solution may still not be an appropriate fuzzy number vector.

Theorem 1: [8] A general solution of a consistent equations SX = Y is $X=S^{-}Y+(I-H)Z$ where $H=S^{-}S$ and Z is an arbitrary vector.

Ezzati's method

Theorem 2: [6] Suppose the inverse of matrix A in Eq. (2) exists and $\tilde{x}=(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n)^T$ is a fuzzy solution of this equation. then

$$\underline{X}+\overline{X}=(\underline{x}_1+\overline{x}_1,\underline{x}_2+\overline{x}_2,\ldots,\underline{x}_n+\overline{x}_n)^T$$

is the solution of the following system $A(\underline{X}+\overline{X})=\underline{y}+\overline{y}$ where

$$\underline{y}+\overline{y}=(\underline{y}_1+\overline{y}_1,\underline{y}_2+\overline{y}_2,\ldots,\underline{y}_n+\overline{y}_n)^T.$$

Ezzati [3]first solved the following system:

$$\begin{cases}a_{11}(\underline{x}_1+\overline{x}_1)+\ldots+a_{1n}(\underline{x}_n+\overline{x}_n)=\underline{y}_1+\overline{y}_1\\ a_{21}(\underline{x}_1+\overline{x}_1)+\ldots+a_{2n}(\underline{x}_n+\overline{x}_n)=\underline{y}_2+\overline{y}_2\\ \quad\vdots\\ a_{n1}(\underline{x}_1+\overline{x}_1)+\ldots+a_{nn}(\underline{x}_n+\overline{x}_n)=\underline{y}_n+\overline{y}_n\end{cases}\qquad (6)$$

and suppose the solution of this system is as

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \underline{\mathbf{X}} + \overline{\mathbf{X}} = \begin{pmatrix} \underline{x}_1 + \overline{x}_1 \\ \underline{x}_2 + \overline{x}_2 \\ \vdots \\ \underline{x}_n + \overline{x}_n \end{pmatrix}$$

then, we have

$$(B+C)\underline{X}(r) = \underline{Y}(r) + Cd \text{ and } (B+C)\overline{X}(r) = \overline{Y}(r) - Cd$$

hence

$$\begin{aligned} \underline{X}(r) &= A^{-1}(\underline{Y}(r) + Cd) \\ \overline{X}(r) &= A^{-1}(\overline{Y}(r) - Cd) \end{aligned} \tag{7}$$

therefore, he can solves fuzzy linear system Eq. (2) by solving Eqs. (6)(7).

THE GENERAL SOLUTIONS OF FUZZY LINEAR SYSTEMS

In this section, we will propose a simply and practical method to solve fuzzy linear systems. At first, we will presenting the following theorem.

Theorem 3: Suppose the $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ is an arbitrary fuzzy solution of Eq. (2). Then the general solutions of the consistent system $A(\underline{X} + \overline{X}) = \underline{Y} + \overline{Y}$ can be expressed by:

$$d = \underline{X} + \overline{X} = A^{-}(\underline{Y} + \overline{Y}) + (I_n - A^{-}A)Z(r)$$

(matrices B and C are defined as section 2.2)
where

$$\underline{Y} + \overline{Y} = (\underline{y}_1 + \overline{y}_1, \underline{y}_2 + \overline{y}_2, \ldots, \underline{y}_n + \overline{y}_n)^T, A^{-}$$

is a g-inverse of matrix A, I_n is an n order unit matrix and Z(r) is an arbitrary vector with parameter r.

Now, we propose a new method for solving the m×n fuzzy linear system. For solving Eq. (2), we first solve the following system:

$$A(\underline{X} + \overline{X}) = \underline{Y} + \overline{Y} \tag{8}$$

and suppose the general solution of this consistent system is as

$$d = \underline{X} + \overline{X} = A^{-}(\underline{Y} + \overline{Y}) + (I_n - A^{-}A)Z(r) \tag{9}$$

Let matrices B and C have defined as section 2.2. Using Eq (3) we get $A\tilde{X} = \tilde{Y}$ or $(B-C)\tilde{X} = \tilde{Y}$ and in parametric form $(B-C)(\underline{X}(r), \overline{X}(r)) = (\underline{Y}(r), \overline{Y}(r))$. We can write this system as follows:

$$\begin{cases} B\underline{X}(r) - C\overline{X}(r) = \underline{Y}(r) \\ B\overline{X}(r) - C\underline{X}(r) = \overline{Y}(r) \end{cases} \tag{10}$$

By substituting of $\overline{X}(r) = d - \underline{X}(r)$ and $\underline{X}(r) = d - \overline{X}(r)$ in the first and second equation of above system, respectively. We have

$$\begin{cases} (B+C)\underline{X}(r) = \underline{Y}(r) + Cd \\ (B+C)\overline{X}(r) = \overline{Y}(r) - Cd \end{cases} \tag{11}$$

now, if the Eq. (11) was a consistent system, then we will have the following general solutions

$$\begin{cases} \underline{X}(r) = (B+C)^{-}(\underline{Y}(r) + Cd) + (I_n - (B+C)^{-}(B+C))Z_1'(r) \\ \overline{X}(r) = (B+C)^{-}(\overline{Y}(r) - Cd) + (I_n - (B+C)^{-}(B+C))Z_2'(r) \end{cases} \tag{12}$$

where $(B+C)^{-}$ is a g-inverse of matrix (B+C), I_n is an n order unit matrix and $Z_1'(r)$ and $Z_2'(r)$ are arbitrary vectors with parameter r.
It is obvious that,

$$\begin{cases} \underline{X}(r) = (B+C)^{-}(\underline{Y}(r) + Cd) + (I_n - (B+C)^{-}(B+C))Z'(r) \\ \overline{X}(r) = d - \underline{X}(r) \end{cases} \tag{13}$$

where $Z'(r)$ is an arbitrary vector with parameter r. Therefore, we can solve consistent fuzzy linear system (2) by solving Eqs. (12) and (13).

Theorem 4: Assume that the maximum numbers of multiplications are required to calculate general solutions for the Eq. (2) by Friedman's method and proposed method are denoted by F_{mn} and E_{mn}, respectively. Then $F_{mn} \geq E_{mn}$ and $F_{mn} - E_{mn} \geq 6mn^2 + 2mn$.

Proof: We know that [8],

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix}$$

where

$$D = \frac{1}{2}[(B+C)^{-1} + (B-C)^{-1}]$$

and

$$E = \frac{1}{2}[(B+C)^{-1} - (B-C)^{-1}]$$

Therefore, for determining S^1, we need to compute $(B+C)^{-1}$ and $(B-C)^{-1}$. Now, assume that N is $n\times n$ matrix and denote by $h_{mn}(N)$ the number of multiplication operations that the required to calculate N^{-1}. It is clear that

$$h_{mn}(S)=h_{mn}(B+C)+h_{mn}(B-C)=2h_{mn}(A)$$

and we know that:

$$\tilde{X} \in E^1 \Rightarrow (\underline{x} = \alpha_1 + \beta_1 r,\ \overline{x} = \alpha_2 + \beta_2 r,\ \alpha_1, \alpha_2, \beta_1, \beta_2 \in R) \quad (14)$$

hence

$$F_{mn} \geq 2h_{mn}(A) + 8mn^2 + 8mn$$

For computing the general solution of Eq. (2) and the general solution for $\underline{X}$ from Eq. (13), the maximum number of multiplication operations are

$$h_{mn}(B+C) + mn^2 + 2mn \text{ and } h_n(A) + mn^2 + 4mn,$$

respectively. Clearly $h_{mn}(B+c) = h_{mn}(A)$, so

$$E_n \geq 2h_{mn}(A) + 2mn^2 + 6mn$$

and hence

$$F_n - E_n \geq 6mn^2 + 2mn$$

Remark 1: It is possible that the Eq (12) is inconsistent or consistent system, when the Eq (2) is consistent system. In this paper, we suppose that the Eq. (2) is consistent system.

EXAMPLES

Example 1: Consider the 3×2 fuzzy linear system

$$\begin{cases} \tilde{x}_1 + \tilde{x}_2 = (0, 1-r) \\ \tilde{x}_1 - \tilde{x}_2 = (0, 1-r) \\ \tilde{x}_1 + 2\tilde{x}_2 = (-1+r, 2-2r) \end{cases} \quad (15)$$

Friedman's method:

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 0 \\ 0 \\ r-1 \\ r-1 \\ r-1 \\ 2r-2 \end{pmatrix}$$

Then, we have

$$S^- = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 1 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0 & 1 & 0 & 0.5 & 0 \end{pmatrix}$$

that it is an arbitrary g-inverse of S, hence the general solutions of the system (15) is

$$X = S^-Y + (I_{2n} - S^-S)Z(r) = \begin{pmatrix} 0.5r-0.5 \\ 2.5r-2.5 \\ 2r-2 \\ 1.5r-1.5 \end{pmatrix} - \begin{pmatrix} 0.5 & 1.5 & 0 & 0.5 \\ 0.5 & -1 & 1.5 & 3 \\ 0 & 0.5 & 0.5 & 1.5 \\ 1.5 & 3 & 0.5 & -1 \end{pmatrix} Z(r)$$

The proposed method:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} \text{ and } B+C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Then, we have

$$A^- = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & -1 \end{pmatrix} \text{ and } (B+C)^- = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$$

we firstly solving Eq(12), so

$$d = \underline{X} + \overline{X} = A^{-}(\underline{Y} + \overline{Y}) + (I_n - A^{-}A)Z(r) = \begin{pmatrix} 1.5 - 1.5r \\ -1 + r \end{pmatrix} - \begin{pmatrix} 0.5 & 1 \\ -1 & -4 \end{pmatrix} Z(r)$$

then, we have the general solutions of the system (15) with the following Eqs:

$$\begin{cases} \underline{X}(r) = (B+C)^{-}(\underline{Y}(r) + Cd) + (I_n - (B+C)^{-}(B+C))Z'(r) = \begin{pmatrix} -1+r \\ -1.5+1.5r \end{pmatrix} + \begin{pmatrix} 0.5 & 2 \\ 0.5 & 2 \end{pmatrix} Z(r) - \begin{pmatrix} 0.5 & 2 \\ 2 & 2 \end{pmatrix} Z'(r) \\ \overline{X}(r) = d - \underline{X}(r) = \begin{pmatrix} 2.5 - 2.5r \\ 0.5 - 0.5r \end{pmatrix} + \begin{pmatrix} -1 & -3 \\ 0.5 & 2 \end{pmatrix} Z(r) + \begin{pmatrix} 0.5 & 2 \\ 2 & 2 \end{pmatrix} Z'(r) \end{cases}$$

Example 2: Consider the 2×3 fuzzy linear system

$$\begin{cases} \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = (r, 2-r) \\ \tilde{x}_1 + \tilde{x}_2 - \tilde{x}_3 = (1+r, 3-r) \end{cases} \tag{16}$$

Friedman's method:

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} r \\ 1+r \\ r-2 \\ r-3 \end{pmatrix} \text{ Then, we have } S^{-} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

that it is an arbitrary g-inverse of S, hence the general solutions of the system is

$$X = S^{-}Y + (I_{2n} - S^{-}S)Z(r) = \begin{pmatrix} 1+2r \\ 1+2r \\ 2r-3 \\ 2r-5 \\ 2r-5 \\ 2r-1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} Z(r)$$

The proposed method:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \text{ and } B + C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \text{ Then, we have } A^{-} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } (B+C)^{-} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

we firstly solving Eq. (12), so

$$d = \underline{X} + \overline{X} = A^{-}(\underline{Y} + \overline{Y}) + (I_n - A^{-}A)Z(r) = \begin{pmatrix} 6 \\ 6 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(r)$$

then, we have the general solutions of the system with the following Eqs:

$$\begin{cases} \underline{X}(r)=(B+C)^{-}(\underline{Y}(r)+Cd)+(I_n-(B+C)^{-}(B+C))Z(r)=\begin{pmatrix} -1+2r \\ -1+2r \\ -1+2r \end{pmatrix}-\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}Z(r)-\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}Z'(r) \\ \overline{X}(r)=d-\underline{X}(r)=\begin{pmatrix} 7-2r \\ 7-2r \\ -1-2r \end{pmatrix}-\begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}Z(r)+\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}Z'(r) \end{cases}$$

CONCLUSION

In the novel proposed method, the original system is replaced by two $m \times n$ crisp linear system. For future work, we try to extend our method to solve a inconsistent fuzzy linear systems.

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