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On the Edge Detour Index Polynomials

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Abstract: The detour index is equal to the sum of the distances between all pairs of vertices of the connected graph on the longest path between corresponding vertices. The edge versions of detour index were defined as the sum of distances between all pairs of edges of the connected graph on the longest path between corresponding edges, recently. We define a generating function, which we call the edge detour index polynomials, whose derivative is the edge detour indices when q = 1. We study some of the properties of these polynomials and compute it for some common graphs.

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INTRODUCTION

The detour matrix is one of the particularly important distance matrices which are based on the topological distance for vertices in a graph. It was introduces into the mathematical literature in 1969 by Frank Harary [1] and it was discussed in 1990 by Buckley and Harary [2]. The detour matrix was introduced into the chemical literature in 1994 under the name "the maximum path matrix of a molecular graph" [3-7]. During these works, the detour index has been defined for a connected graph G as follows:

$$D(G) = \sum_{\{u,v\}\subseteq V(G)} \Delta(u,v)$$

where $\Delta(u,v)$ denotes the detour distance which is the distance between the vertices *u* and *v* on the longest path.

The detour index polynomial of a graph G was introduced very recently as follows [8]. If q is a parameter, then the detour index polynomial of G is

$$D(G;q) = \sum_{\{x,y\}\subseteq V(G)} q^{\Delta(x,y)}$$

Also, the edge versions of detour index are the sum of distances between edges of a connected graph G on the longest path as follow [9]:

The first edge-detour index is:

$$D_{e0}(G) = \sum_{\{e,f\}\in E(G)} \Delta_0(e,f) = \sum_{\{e,f\}\in V(L(G))} \Delta_0(e,f)$$

where $\Delta_0(u,v)$ is the detour index in Line graph L(G).

Here, the line graph L(G) is the intersection graph of the edges of G, where vertices correspond to edges of G and vertices in L (G) are adjacent if the corresponding edges share a vertex.

The second edge-detour index is:

$$D_{e3}(G) = \sum_{\{e,f\}\in E(G)} \Delta_3(e,f)$$

Where

$$\Delta_3(e,f) = \begin{cases} \Delta_1(e,f) + 1 & , e \neq f \\ 0 & , e = f \end{cases}$$

and

$$\Delta_1(e,f) = \min\{\Delta(u,x), \Delta(u,y), \Delta(v,x), \Delta(v,y)\}$$

where e = uv and *f* = xy.

The third edge-detour index is:

$$D_{e4}(G) = \sum_{\{e,f\}\in E(G)} \Delta_4(e,f)$$

Where

$$\Delta_4(e,f) = \begin{cases} \Delta_2(e,f) & , e \neq f \\ 0 & , e = f \end{cases}$$

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and

$$\Delta_2(e,f) = \max\{\Delta(u,x), \Delta(u,y), \Delta(v,x), \Delta(v,y)\}$$

where e = uv and f = xy.

In this paper, we define the polynomials of edge versions of detour index. And we study some of the properties of these polynomials and compute it for some well known graphs.

SOME DEFINITIONS AND RESULTS

We wish to define and study the edge detour index polynomial in this section.

Definition 2-1: If q is a parameter, then the edge detour index polynomials of a graph G is

$$D_{ei}(G;q) = \sum_{\{e,f\}\in E(G)} q^{\Delta_i(e,f)}$$

where i = 0,3,4

In what follows, we use |S| to describe the cardinal of a set S. Also, if f(q) is a polynomial in q,

$$F = \left\{ \{e,f\} \subseteq E(G) \middle| \begin{array}{l} \text{theedgeefisnoton} \\ \text{any cycleingraph } L(G) \end{array} \right\}$$

$$T = \left\{ \{e,f\} \subseteq E(G) \middle| \begin{array}{l} \text{theedgeseandfareonly} \\ \text{ontriangleingraph } L(G) \end{array} \right\}$$

then deg$f$(g) is its degree and $[q^i]f(g)$ is the coefficient of q^i.

The next theorem summarizes some of the properties of D_{ei} (G;g). Its proof follows easily from the definitions and so is omitted.

Theorem 2-2: Suppose i = 0,3,4, we have:

1. deg D_{ei} (G;g) = diameter of G under distance Δ_i.
2. $[q^0]D_{ei}(G;q) = 0$
3. $[q^1]D_{e3}(G;q) = |F|$, $[q^1]D_{e4}(G;q) = |T|$ and $[q^1]D_{e0}(G;q) = |F|$
4. $D_{ei}(G;1) = \binom{|E(G)|}{2}$
5. $D_{ei}{}'(G;1) = D_{ei}(G)$

Now, we compute the edge detour index polynomials of some familiar graphs K_n, $K_{m,n}$ and C_n which are complete graph, complete bipartite graph and cycle, respectively. Before our computations we state some results and relations between distances Δ_0, Δ_3 and Δ_4 for graphs K_n, $K_{m,n}$ and C_n in following.

In Fig. 1, the graph K_g with the Hamiltonian cycle on its line graph is shown.

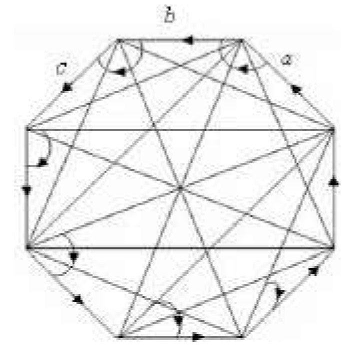

Fig. 1: algorithm for finding the Hamiltonian cycle of $L(K_n)$

The algorithm for finding the Hamiltonian cycle of $L(K_n)$ due to Fig. 1 is:

1. Fix an edge which is placed on outer cycle of K_n. We name this edge with a.
2. Move from a to its neighbor edge b on outer cycle of K_n by passing its other neighbors which are common with b.
3. Move from b to its neighbor edge c on outer cycle of K_n by passing its other neighbors which we did not pass them during our moving.
4. Do the procedure 3 for edge c and other edges which are in outer cycle of K_n and we reach them during our moving.
5. Finally we reach to edge a after passing from all edges of K_n. Therefore this cycle is Hamiltonian

Therefore, according to above explanations we have the following result.

Result 1: The line graph of K_n which is shown with notation $L(K_n)$ is Hamiltonian.

In Fig. 2, the graph K_8 with the Hamiltonian path between each pair of edges of K_8 on its line graph is shown. The case (i) shows the Hamiltonian path between two edge which are on outer cycle of K_8, the case (ii) shows the Hamiltonian path between two edge which are not on outer cycle of K_8 and the case (iii) shows the Hamiltonian path between two edge which one edge is on outer cycle of K_8 and another is not.

The algorithm for finding the Hamiltonian path of $L(K_n)$ due to Fig. 2-(i) is, n≥4:

1. Fix two nonadjacent edges which is placed on outer cycle of K_n. We name these edges with a and b.

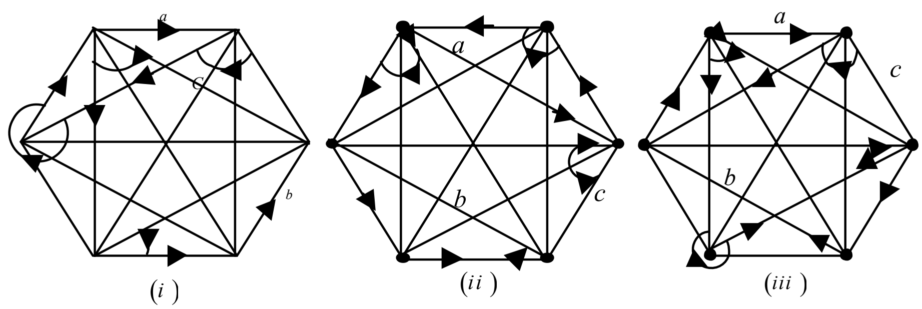

Fig. 2: The graph K_8 with the Hamiltonian path

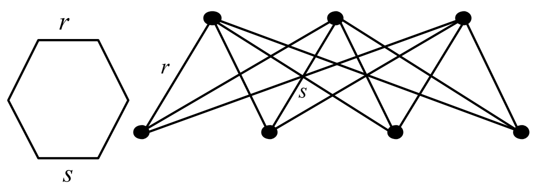

Fig. 3: Hamiltonian path

2. Move from a to its neighbor edge c.
3. Move from c to another edge which is on outer cycle of K_n and is neighbor of a by passing the neighbor edges of a.
4. Do the procedure 3 for the last neighbor edge of a and other edges which are in outer cycle of K_n and we reach them during our moving to edge b.
5. Finally we reach form a to edge b on Hamiltonian path.
6. Therefore we can find the Hamiltonian between each pairs of edges.

Therefore, according to above explanations we have the following result.

Result 2: There exists a Hamiltonian path between each pair of edges of K_n (vertices of $L(K_n)$) in $L(K_n)$.

In what follows, the parallel edges in complete bipartite graphs and cycles is the edges which are parallel geometrically. For example, the edges r and s in following edges are parallel.

Claim 1: The line graph of complete bipartite graph $L(K_{m,n})$ is Hamiltonian.

Proof: It is enough that we find a Hamiltonian cycle in $L(K_{m,n})$. For example, consider Fig. 4.

By the algorithm which is mentioned in Fig. 4 we get the desire result. This algorithm is:

Consider a complete bipartite graph $K_{m,n}$ which consisted of parts of A and B.

1. Fix an edge a = uv which $u \in A$ and $v \in B$.
2. Move from a to its neighbor edges which is common in vertex u.

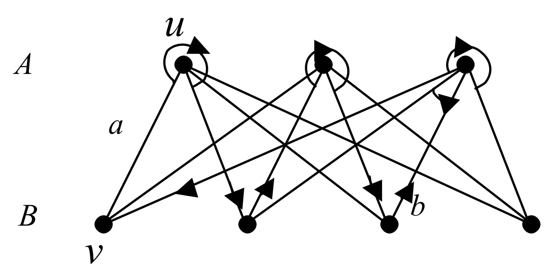

Fig. 4: Bipartite graph $K_{m,n}$

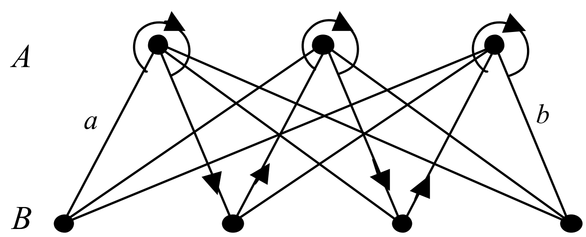

Fig. 5: The vertices of $K_{m,n}$ that a and b are the outer edges of $K_{m,n}$ as shown

3. Go from the last neighbor edge to the nonadjacent parallel edge of a.
4. Move from mentioned edge to its neighbor edges which is common in vertex that located in part A.
5. Do the processes of 4 and 5 before reaching to the last nonadjacent parallel edge b with a.
6. Go from mentioned edge b to a by passing the neighbor edges of b which have common vertex in part A.
7. Therefore, by above algorithm we can find a Hamiltonian cycle.

Claim 2: There exists a Hamiltonian path between each pair of edges of $K_{m,n}$ (vertices of $L(K_{m,n})$).

Proof: According to the claim 3, two edges which are adjacent the result is clear. Then, we prove the result for nonadjacent edges. Consider the Fig. 5.

The algorithm which is mentioned in Fig. 5, is:

1. Consider a complete bipartite graph $K_{m,n}$ which consisted of parts of A and B.
2. Fix two nonadjacent edges a and b.

3. rearrange the vertices of $K_{m,n}$ that a and b are the outer edges of $K_{m,n}$ as shown in Fig. 5.
4. Move from a to its neighbor edges which are common in vertex which is in part A.
5. Go from the last neighbor edge to the first parallel edge with a and do the process 4 for this edge.
6. Do the processes 4 and 5 before reaching to the edge b.

Therefore, we conclude the desire result.

Lemma 2-3: Let K_n be the complete graph with n vertex and $e,f \in E(K_n)$. Therefore we have:

$$\Delta_0(e,f) = \binom{n}{2} - 1, \Delta_3(e,f) = n$$

and $\Delta_4(e,f) = n-1$.

Proof: Suppose K_n is the complete graph with n vertices and $e,f \in E(K_n)$.

For computation the distance Δ_0, we must see the line graph $L(K_n)$ and according to the Claims 1 and 2, it is clear that

$$\Delta_0(e,f) = \binom{n}{2} - 1$$

For computation the distance Δ_3 and Δ_4, we must see the graph K_n. According to the facts that graph K_n is Hamiltonian and there exist a Hamiltonian path between each pair of vertices, $\Delta_3(e,f) = n$ and $\Delta_4(e,f) = n-1$.

Lemma 2-4: Let $K_{m,n}$ be the complete bipartite graph which constructed by two set A and B such that $|A| = n$ and $|B| = m$, $m \le n$ and $e,f \in E(K_{m,n})$. Therefore, we have:

1. $\Delta_0(e,f) = mn - 1$
2. i) If $m<n$, then

$$\Delta_3(e,f) = \begin{cases} 2m-1 & , \quad e,f \text{ are nonadjacent} \\ 2m & , \quad e,f \text{ are adjacent and have common vertex in set B} \\ 2m-1 & , \quad e,f \text{ are adjacent and have common vertex in set A} \end{cases}$$

ii) If $m = n$, then $\Delta_3(e,f) = 2m - 1$

3. i) If $m<n$, then

$$\Delta_4(e,f) = \begin{cases} 2m & , \quad e,f \text{ are nonadjacent} \\ 2m & , \quad e,f \text{ are adjacent and have common vertex in set B} \\ 2m-1 & , \quad e,f \text{ are adjacent and have common vertex in set A} \end{cases}$$

ii) If $m = n$, then $\Delta_4(e,f) = 2m - 1$

Proof: Suppose $K_{m,n}$ be the complete bipartite graph, $m \le n$ and $e,f \in E(K_{m,n})$.

For computation the distance Δ_0, we must see the line graph $L(K_{m,n})$ and according to the Claims 3 and 4, it is clear that $\Delta_0(e,f) = mn-1$.

For computation the distance Δ_3 and Δ_4, we must see the graph $K_{m,n}$. According to the facts that the length of longest cycle is 2m in $K_{m,n}$ and there is or is not Hamiltonian path between nonadjacent vertices, we can conclude the desire results about $\Delta_3(e,f)$ and $\Delta_4(e.f)$.

In following results we use some notations for simplifying our computations. These notations are:

$$C_1 = \{\{e,f\} \in E(C_{2n}) | \Delta_0(e,f) = n\}$$

and

$$C_2 = \{\{e,f\} \in E(C_{2n+1}) | \Delta_0(e,f) = n+1\} .$$

Lemma 2-5: Let C_n be the cycle and $e,f \in E(C_n)$. Therefore we have:

$$\Delta_0(e,f) = \begin{cases} \Delta_3(e,f) - 1 = \Delta_4(e,f) - 1 & , \quad \{e,f\} \in C_1 \text{ or } C_2 \\ \Delta_3(e,f) = \Delta_4(e,f) & , \quad e,f \text{ is adjecent} \\ \Delta_3(e,f) = \Delta_4(e,f) - 1 & , \quad \text{o. w.} \end{cases} .$$

Proof: It is easily concluded from the shape of the cycle C_n.

In following we state some results according to above Lemmas.

Result 3: Let K_n be the complete graph with n vertex. Then,

$$D_{e0}(K_n) = \frac{(n+1)(n-2)}{2n} D_{e3}(K_n) = \frac{(n+1)(n-2)}{2(n-1)} D_{e4}(K_n) = \binom{\binom{n}{2}}{2}\left(\binom{n}{2} - 1\right) .$$

Result 4: Let $K_{m,n}$ be the complete bipartite graph and $m \le n$. Then,

$$D_{e0}(K_{m,n}) = mn(mn-1) = \begin{cases} \frac{(mn-1)}{2m-1}\left(D_{e3}(K_{m,n}) - m\binom{n}{2}\right) = \frac{(mn-1)}{2m}\left(D_{e4}(K_{m,n}) + n\binom{m}{2}\right) & ,m \neq n \\ \frac{(mn-1)}{2m-1}D_{e3}(K_{m,n}) = \frac{(mn-1)}{2m-1}D_{e4}(K_{m,n}) & ,m = n \end{cases}.$$

Result 5: Let C_n be the cycle. Then,

$$D_{e0}(C_n) = \begin{cases} D_{e3}(C_n) - \frac{n}{2} = D_{e4}(C_n) - \left(\binom{n}{2} - n\right) & ,n \text{ is even} \\ D_{e3}(C_n) - n = D_{e4}(C_n) - \left(\binom{n}{2} - n\right) & ,n \text{ is odd} \end{cases}$$

At following we state edge detour index polynomials of K_n, $K_{m,n}$ and C_n.

Theorem 2-6: Let K_n, $K_{m,n}$ and C_n be the complete, complete bipartite graphs and cycle, respectively and suppose $m \leq n$. Then, we have:

1. $D_{e0}(K_n;q) = \binom{|E(K_n)|}{2} q^{\left(\binom{n}{2}-1\right)}$, $D_{e3}(K_n;q) = \binom{|E(K_n)|}{2} q^{n}$

and

$$D_{e4}(K_n;q) = \binom{|E(K_n)|}{2} q^{(n+1)}$$

2. $D_{e0}(K_{m,n};q) = \binom{|E(K_{m,n})|}{2} q^{(mn-1)}$

$$D_{e3}(K_{m,n};q) = \begin{cases} q^{2m-1}\left(\binom{mn}{2} + m\binom{n}{2}(q-1)\right) & ,m \neq n \\ \binom{mn}{2} q^{(2m-1)} & ,m = n \end{cases}$$

and

$$D_{e4}(K_{m,n};q) = \begin{cases} q^{2m-1}\left(\binom{mn}{2} + n\binom{m}{2}(1-q)\right) & ,m \neq n \\ \binom{mn}{2} q^{(2m-1)} & ,m = n \end{cases}$$

3. $$D_{e0}(C_n;q) = \begin{cases} q^{\frac{n}{2}}\left(q + q^2 + \ldots + q^{\frac{n}{2}-2} + n\left(\frac{1}{2} + q^{\frac{n}{2}-1}\right)\right) & ,n \text{ is even} \\ q^{\frac{n+1}{2}}\left(1 + q + \ldots + q^{\frac{n-7}{2}} + n\left(1 + q^{\frac{n-3}{2}}\right)\right) & ,n \text{ is odd} \end{cases}$$

$$D_{e3}(C_n;q) = \begin{cases} q^{\frac{n}{2}}\left(q + q^2 + \ldots + q^{\frac{n}{2}-2} + n\left(\frac{1}{2}q^{-1} + q^{\frac{n}{2}-1}\right)\right) & ,n \text{ is even} \\ q^{\frac{n+1}{2}}\left(1 + q + \ldots + q^{\frac{n-7}{2}} + n\left(q^{-1} + q^{\frac{n-3}{2}}\right)\right) & ,n \text{ is odd} \end{cases}$$

and

$$D_{e4}(C_n;q) = \begin{cases} q^{\frac{n}{2}}\left(q + q^2 + \ldots + q^{\frac{n}{2}-2} + n\left(\frac{1}{2}q^{-1} + q^{\frac{n}{2}-1}\right)\right) & ,n \text{ is even} \\ q^{\frac{n+1}{2}}\left(1 + q + \ldots + q^{\frac{n-7}{2}} + n\left(q^{-1} + q^{\frac{n-3}{2}}\right)\right) & ,n \text{ is odd} \end{cases}.$$

Proof: Due to the Lemmas (2-3, 2-4 and 2-5), the desire results in 1 and 2 can be concluded. Therefore, we prove only the third result. We have from the definition of edge detour index polynomials for C_n:

$$D_{e0}(C_n;q) = \begin{cases} \frac{n}{2}q^{\frac{n}{2}} + nq^{(n-1)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_1 \\ e,f \text{ are nonadjacent}}} q^{\Delta_0(e,f)} & ,n \text{ is even} \\ nq^{\frac{n+1}{2}} + nq^{(n-1)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_2 \\ e,f \text{ are nonadjacent}}} q^{\Delta_0(e,f)} & ,n \text{ is odd} \end{cases}$$

$$D_{e3}(C_n;q) = \begin{cases} \frac{n}{2}q^{\frac{n}{2}-1} + nq^{(n-1)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_1 \\ e,f \text{ are nonadjacent}}} q^{\Delta_0(e,f)} & ,n \text{ is even} \\ nq^{\frac{n-1}{2}} + nq^{(n-1)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_2 \\ e,f \text{ are nonadjacent}}} q^{\Delta_0(e,f)} & ,n \text{ is odd} \end{cases}$$

and

$$D_{e4}(C_n;q) = \begin{cases} \frac{n}{2}q^{\frac{n}{2}-1} + nq^{(n-1)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_1 \\ e,f \text{ are nonadjacent}}} q^{\Delta_0(e,f)} & ,n \text{ is even} \\ nq^{\frac{n-1}{2}} + nq^{(n+1)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_2 \\ e,f \text{ are nonadjacent}}} q^{\Delta_0(e,f)} & ,n \text{ is odd} \end{cases}.$$

Also we have

$$\sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_1 \\ e,f \text{ are nonadjacent}}} q^{\Delta_0(e,f)} = q^{\frac{n}{2}}\left(q + q^2 + \ldots + q^{\frac{n}{2}-2}\right)$$

and

$$\sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_2 \\ e,f \text{ are nonadjacent}}} q^{\Delta_0(e,f)} = q^{\frac{n+1}{2}}\left(1 + q + \ldots + q^{\frac{n-7}{2}}\right)$$

Then with simplifying the edge detour index polynomials of C_n, the desire result can be concluded.

Result 6: The relations between edge detour index polynomials of K_n, $K_{m,n}$ which m≤n and C_n are as follows:

1. $D_{e0}(K_n;q) = q^{\frac{n^2-3n-2}{2}} D_{e3}(K_n;q) = q^{\frac{n^2-3n}{2}} D_{e4}(K_n;q)$

2. $D_{e0}(K_n;q) = \begin{cases} q^{m(n-2)} D_{e3}(K_n;q) - m\binom{n}{2} q^{m\,n-1}(q-1) & ,m \neq n \\ q^{(m\,n-2\,m)} D_{e3}(K_n;q) & ,m = n \end{cases}$ and $D_{e0}(K_n;q) = \begin{cases} q^{(m\,n-2\,m-1)} D_{e4}(K_n;q) - n\binom{m}{2} q^{m\,n-2}(1-q) & ,m \neq n \\ q^{(m\,n-2\,m)} D_{e4}(K_n;q) & ,m = n \end{cases}$

3. $D_{e4}(C_n;q) = qD_{e0}(C_n;q) + n(1-q)q^{n-1} = \begin{cases} qD_{e3}(C_n;q) + (1-q)\left(\frac{n}{2} q^{\frac{n}{2}+1} + nq^{n-1}\right) & ,n\,is\,even \\ qD_{e3}(C_n;q) + (1-q)\left(nq^{\frac{n+1}{2}+1} + nq^{n-1}\right) & ,n\,is\,odd \end{cases}$

Proof: Consider the graphs K_n, $K_{m,n}$ which m≤n and C_n. Therefore,

1. For graph K_n we have

$$D_{e0}(K_n;q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_0(e,f)} = \binom{|E(K_n)|}{2} q^{\left(\binom{n}{2}-1\right)} = q^{\frac{n^2-3n-2}{2}} \binom{|E(K_n)|}{2} q^{n} = q^{\frac{n^2-3n-2}{2}} D_{e3}(K_n;q)$$

$$D_{e0}(K_n;q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_0(e,f)} = \binom{|E(K_n)|}{2} q^{\left(\binom{n}{2}-1\right)} = q^{\frac{n^2-3n}{2}} \binom{|E(K_n)|}{2} q^{(n-1)} = q^{\frac{n^2-3n}{2}} D_{e4}(K_n;q)$$

Then, with easy computations, the desire results in (1) can be concluded.

2. For graph $K_{m,n}$ which m≤n, we have

$$D_{e0}(K_{m,n};q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_0(e,f)} = \binom{mn}{2} q^{mn-1}$$

and

i) If m<n, then

$$D_{e3}(K_{m,n};q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_3(e,f)} = \binom{mn}{2} q^{2m-1} + m\binom{n}{2} q^{2m-1}(q-1)$$

and

$$D_{e4}(K_{m,n};q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_4(e,f)} = \binom{mn}{2} q^{2m} + n\binom{m}{2} q^{2m-1}(q-1)$$

ii) If m = n, then

$$D_{e3}(K_{m,n};q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_3(e,f)} = \binom{mn}{2} q^{2m-1} \text{ and } D_{e4}(K_{m,n};q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_4(e,f)} = \binom{mn}{2} q^{2m-1}.$$

Then, with easy computations, the desire results in (2) can be concluded.

3. For graph C_n we have

$$D_{e4}(C_n;q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_4(e,f)} = \sum_{\{e,f\}\in C_1\,or\,C_2} q^{\Delta_4(e,f)} + \sum_{e,f\,are\,adjacent} q^{\Delta_4(e,f)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\notin C_1\,and\,C_2 \\ e,f\,are\,nonadjacent}} q^{\Delta_4(e,f)}$$

$$= \sum_{\{e,f\}\in C_1\,or\,C_2} q^{\Delta_0(e,f)+1} + \sum_{e,f\,are\,adjacent} q^{\Delta_0(e,f)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\notin C_1\,and\,C_2 \\ e,f\,are\,nonadjacent}} q^{\Delta_0(e,f)+1}$$

$$= q\sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_0(e,f)} + (1-q)\sum_{e,f\,are\,adjacent} q^{\Delta_0(e,f)} = qD_{e0}(C_n;q) + n(1-q)q^{(n-1)}$$

$$D_{e4}(C_n;q) = \sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_4(e,f)} = \sum_{\{e,f\}\subseteq C_1 \text{ or } C_2} q^{\Delta_4(e,f)} + \sum_{e,f \text{ are adjacent}} q^{\Delta_4(e,f)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_1 \text{ and } C_2 \\ e,f \text{ are nonadjacent}}} q^{\Delta_4(e,f)}$$

$$= \sum_{\{e,f\}\subseteq C_1 \text{ or } C_2} q^{\Delta_3(e,f)} + \sum_{e,f \text{ are adjacent}} q^{\Delta_3(e,f)} + \sum_{\substack{\{e,f\}\subseteq E(C_n) \\ \{e,f\}\not\subseteq C_1 \text{ and } C_2 \\ e,f \text{ are nonadjacent}}} q^{\Delta_3(e,f)+1}$$

$$= q\sum_{\{e,f\}\subseteq E(C_n)} q^{\Delta_3(e,f)} + (1-q)\left(\sum_{e,f \text{ are adjacent}} q^{\Delta_0(e,f)} + \sum_{\{e,f\}\in C_1 \text{ or } C_2} q^{\Delta_3(e,f)}\right)$$

$$= \begin{cases} qD_{e3}(C_n;q) + (1-q)\left(\frac{n}{2}q^{\frac{n}{2}+1} + nq^{n-1}\right) & , n \text{ is even} \\ qD_{e3}(C_n;q) + (1-q)\left(nq^{\frac{n+1}{2}+1} + nq^{n-1}\right) & , n \text{ is odd} \end{cases}$$

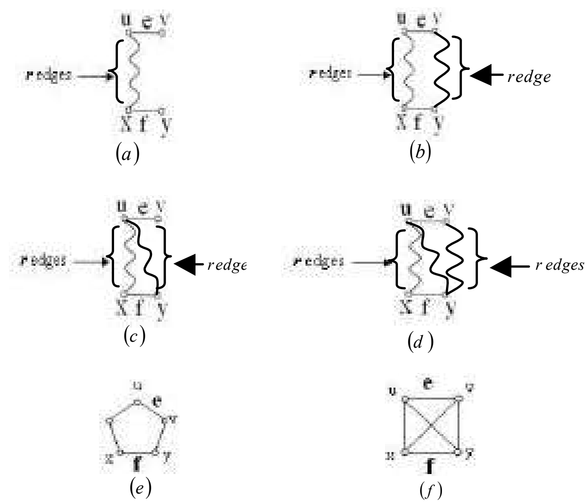

Fig. 6: The quantity $\Delta'_P(e, f)$ is r for shapes (a,b,c and d) and is 1 for shapes (e and f).

Then, with easy computations, the desire results in (3) can be concluded.

EDGE DETOUR INDEX POLYNOMIALS DUE TO DISTANCES BETWEEN VERTICES

In following, we restate the edge detour index polynomials according to distances between vertices. For finding these polynomials, we must find the relations among distances between edges with distances between vertices.

Definition 3-1: Let e, $f \in E(G)$, e = uv and f = xy. Fix a longest path between edges e and f and name it P. We define the quantity $\Delta'_P(e, f)$ as follows.

$$\Delta'_P(e,f) = \min\{\Delta_P(u,x), \Delta_P(u,y), \Delta_P(v,x), \Delta_P(v,y)\}$$

where Δ_P is length of distances of vertices on path P.

Due to the fact that edge detour distances are defined on longest path between edges, we can imagine six case for two edge and longest paths between them. These six case are shown in following Fig. 6.

Therefore, we partite the set of pair edges to following subsets.

$$A_1 = \{\{e,f\}\subseteq E(G) \mid e,f \text{ are edges the same as Figure 6(a)}\}$$

$$A_2 = \{\{e,f\}\subseteq E(G) \mid e,f \text{ are edges the same as Figure 6(b)}\}$$

$$A_3 = \{\{e,f\}\subseteq E(G) \mid e,f \text{ are edges the same as Figure 6(c)}\}$$

$$A_4 = \{\{e,f\}\subseteq E(G) \mid e,f \text{ are edges the same as Figure 6(d)}\}$$

$$A_5 = \{\{e,f\}\subseteq E(G) \mid e,f \text{ are edges the same as Figure 6(e)}\}$$

$$A_6 = \{\{e,f\}\subseteq E(G) \mid e,f \text{ are edges the same as Figure 6(f)}\}$$

Then, we find the edge detour distances as follows:

Lemma 3-2: Let $e,f \in E(G)$, e = uv and f = xy. Then,

$$\Delta_0(e,f) = \begin{cases} \Delta'(e,f)+1 & ,\{e,f\}\in A_1 \\ \Delta'(e,f)+1 & ,\{e,f\}\in A_2 \\ \Delta'(e,f)+2 & ,\{e,f\}\in A_3 \\ 3\Delta'(e,f)+1 & ,\{e,f\}\in A_4 \\ \Delta'(e,f)+2 & ,\{e,f\}\in A_5 \\ 4\Delta'(e,f)+1 & ,\{e,f\}\in A_6 \end{cases}$$

$$\Delta_3(e,f) = \begin{cases} \Delta'(e,f)+1 & ,\{e,f\}\in A_1 \\ \Delta'(e,f)+2 & ,\{e,f\}\in A_2 \\ \Delta'(e,f)+2 & ,\{e,f\}\in A_3 \\ \Delta'(e,f)+2 & ,\{e,f\}\in A_4 \\ \Delta'(e,f)+3 & ,\{e,f\}\in A_5 \\ 3\Delta'(e,f)+1 & ,\{e,f\}\in A_6 \end{cases}$$

and

$$\Delta_4(e,f)=\begin{cases}\Delta'(e,f)+2 & ,\{e,f\}\in A_1\\ \Delta'(e,f)+2 & ,\{e,f\}\in A_2\\ \Delta'(e,f)+2 & ,\{e,f\}\in A_3\\ 3\Delta'(e,f) & ,\{e,f\}\in A_4\\ \Delta'(e,f)+3 & ,\{e,f\}\in A_5\\ 3\Delta'(e,f) & ,\{e,f\}\in A_6\end{cases}$$

Proof: From the Definition (3-1), subsets of set of pair edges and Fig. 6, we can get the desire results.

Theorem 3-3: The relations between edge detour indices are:

$$D_{e0}(G)-D_{e3}(G)=2\sum_{\{e,f\}\in A_4}\Delta'(e,f)+\sum_{\{e,f\}\in A_6}\Delta'(e,f)-|A_2|-|A_4|-|A_5|$$

and

$$D_{e0}(G)-D_{e4}(G)=\sum_{\{e,f\}\in A_6}\Delta'(e,f)-|A_1|-|A_2|+|A_4|-|A_5|+|A_6|$$

Proof: Due to the definitions of edge detour indices, Definition (3-1) and Lemma (3-2), we have:

$$\begin{aligned}D_{e0}(G)&=\sum_{\{e,f\}\subseteq E(G)}\Delta_0(e,f)=\sum_{\{e,f\}\in A_1}\Delta_0(e,f)+\sum_{\{e,f\}\in A_2}\Delta_0(e,f)+\sum_{\{e,f\}\in A_3}\Delta_0(e,f)+\sum_{\{e,f\}\in A_4}\Delta_0(e,f)+\sum_{\{e,f\}\in A_5}\Delta_0(e,f)+\sum_{\{e,f\}\in A_6}\Delta_0(e,f)\\&=\sum_{\{e,f\}\in A_1}(\Delta'(e,f)+1)+\sum_{\{e,f\}\in A_2}(\Delta'(e,f)+1)+\sum_{\{e,f\}\in A_3}(\Delta'(e,f)+2)\\&+\sum_{\{e,f\}\in A_4}(3\Delta'(e,f)+1)+\sum_{\{e,f\}\in A_5}(\Delta'(e,f)+2)+\sum_{\{e,f\}\in A_6}(4\Delta'(e,f)+1)\\&=\sum_{\{e,f\}\subseteq E(G)}\Delta'(e,f)+\sum_{\{e,f\}\in A_4}2\Delta'(e,f)+\sum_{\{e,f\}\in A_6}3\Delta'(e,f)+|A_1|+|A_2|+2|A_3|+|A_4|+2|A_5|+|A_6|\end{aligned}$$

and

$$\begin{aligned}D_{e3}(G)&=\sum_{\{e,f\}\subseteq E(G)}\Delta_3(e,f)=\sum_{\{e,f\}\in A_1}\Delta_3(e,f)+\sum_{\{e,f\}\in A_2}\Delta_3(e,f)+\sum_{\{e,f\}\in A_3}\Delta_3(e,f)+\sum_{\{e,f\}\in A_4}\Delta_3(e,f)+\sum_{\{e,f\}\in A_5}\Delta_3(e,f)+\sum_{\{e,f\}\in A_6}\Delta_3(e,f)\\&=\sum_{\{e,f\}\in A_1}(\Delta'(e,f)+1)+\sum_{\{e,f\}\in A_2}(\Delta'(e,f)+2)+\sum_{\{e,f\}\in A_3}(\Delta'(e,f)+2)\\&+\sum_{\{e,f\}\in A_4}(\Delta'(e,f)+2)+\sum_{\{e,f\}\in A_5}(\Delta'(e,f)+3)+\sum_{\{e,f\}\in A_6}(3\Delta'(e,f)+1)\\&=\sum_{\{e,f\}\subseteq E(G)}\Delta'(e,f)+\sum_{\{e,f\}\in A_6}2\Delta'(e,f)+|A_1|+2|A_2|+2|A_3|+2|A_4|+3|A_5|+|A_6|\end{aligned}$$

and

$$\begin{aligned}D_{e4}(G)&=\sum_{\{e,f\}\subseteq E(G)}\Delta_4(e,f)=\sum_{\{e,f\}\in A_1}\Delta_4(e,f)+\sum_{\{e,f\}\in A_2}\Delta_4(e,f)+\sum_{\{e,f\}\in A_3}\Delta_4(e,f)+\sum_{\{e,f\}\in A_4}\Delta_4(e,f)+\sum_{\{e,f\}\in A_5}\Delta_4(e,f)+\sum_{\{e,f\}\in A_6}\Delta_4(e,f)\\&=\sum_{\{e,f\}\in A_1}(\Delta'(e,f)+2)+\sum_{\{e,f\}\in A_2}(\Delta'(e,f)+2)+\sum_{\{e,f\}\in A_3}(\Delta'(e,f)+2)\\&+\sum_{\{e,f\}\in A_4}(3\Delta'(e,f))+\sum_{\{e,f\}\in A_5}(\Delta'(e,f)+3)+\sum_{\{e,f\}\in A_6}(3\Delta'(e,f))\\&=\sum_{\{e,f\}\subseteq E(G)}\Delta'(e,f)+\sum_{\{e,f\}\in A_4}2\Delta'(e,f)+\sum_{\{e,f\}\in A_6}2\Delta'(e,f)+2|A_1|+2|A_2|+2|A_3|+3|A_5|\end{aligned}$$

Hence, the results can be concluded only with computing the differences of $D_{e0}(G)$, $D_{e3}(G)$ and $D_{e4}(G)$.

Theorem 3-4: The relations between edge-detour index polynomials are:

$$D_{e0}(G;q) - D_{e3}(G;q) = (1-q)\sum_{\{e,f\}\in A_2} q^{(\Delta'(e,f)+1)} + \sum_{\{e,f\}\in A_4} q^{(\Delta'(e,f)+2)}\left(q^{(2\Delta'(e,f)-1)}-1\right)$$
$$+(1-q)\sum_{\{e,f\}\in A_5} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_6} q^{(3\Delta'(e,f)+1)}\left(q^{\Delta'(e,f)}-1\right).$$

and

$$D_{e0}(G;q) - D_{e4}(G;q) = (1-q)\sum_{\{e,f\}\in A_1} q^{(\Delta'(e,f)+1)} + (1-q)\sum_{\{e,f\}\in A_2} q^{(\Delta'(e,f)+1)}$$
$$+(q-1)\sum_{\{e,f\}\in A_4} q^{(3\Delta'(e,f))} + (1-q)\sum_{\{e,f\}\in A_5} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_6} q^{(3\Delta'(e,f))}\left(q^{(\Delta'(e,f)+1)}-1\right)$$

Proof: Due to the definitions of edge detour index polynomials, Definition (3-1) and Lemma (3-2), we have:

$$D_{e0}(G;q) = \sum_{\{e,f\}\subseteq E(G)} q^{\Delta_0(e,f)} = \sum_{\{e,f\}\in A_1} q^{\Delta_0(e,f)} + \sum_{\{e,f\}\in A_2} q^{\Delta_0(e,f)} + \sum_{\{e,f\}\in A_3} q^{\Delta_0(e,f)} + \sum_{\{e,f\}\in A_4} q^{\Delta_0(e,f)} + \sum_{\{e,f\}\in A_5} q^{\Delta_0(e,f)} + \sum_{\{e,f\}\in A_6} q^{\Delta_0(e,f)}$$
$$= \sum_{\{e,f\}\in A_1} q^{(\Delta'(e,f)+1)} + \sum_{\{e,f\}\in A_2} q^{(\Delta'(e,f)+1)} + \sum_{\{e,f\}\in A_3} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_4} q^{(3\Delta'(e,f)+1)} + \sum_{\{e,f\}\in A_5} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_6} q^{(4\Delta'(e,f)+1)}$$

$$D_{e3}(G;q) = \sum_{\{e,f\}\subseteq E(G)} q^{\Delta_3(e,f)} = \sum_{\{e,f\}\in A_1} q^{\Delta_3(e,f)} + \sum_{\{e,f\}\in A_2} q^{\Delta_3(e,f)} + \sum_{\{e,f\}\in A_3} q^{\Delta_3(e,f)} + \sum_{\{e,f\}\in A_4} q^{\Delta_3(e,f)} + \sum_{\{e,f\}\in A_5} q^{\Delta_3(e,f)} + \sum_{\{e,f\}\in A_6} q^{\Delta_3(e,f)}$$
$$= \sum_{\{e,f\}\in A_1} q^{(\Delta'(e,f)+1)} + \sum_{\{e,f\}\in A_2} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_3} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_4} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_5} q^{(\Delta'(e,f)+3)} + \sum_{\{e,f\}\in A_6} q^{(3\Delta'(e,f)+1)}$$

$$D_{e4}(G;q) = \sum_{\{e,f\}\subseteq E(G)} q^{\Delta_4(e,f)} = \sum_{\{e,f\}\in A_1} q^{\Delta_4(e,f)} + \sum_{\{e,f\}\in A_2} q^{\Delta_4(e,f)} + \sum_{\{e,f\}\in A_3} q^{\Delta_4(e,f)} + \sum_{\{e,f\}\in A_4} q^{\Delta_4(e,f)} + \sum_{\{e,f\}\in A_5} q^{\Delta_4(e,f)} + \sum_{\{e,f\}\in A_6} q^{\Delta_4(e,f)}$$
$$= \sum_{\{e,f\}\in A_1} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_2} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_3} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_4} q^{(3\Delta'(e,f))} + \sum_{\{e,f\}\in A_5} q^{(\Delta'(e,f)+3)} + \sum_{\{e,f\}\in A_6} q^{(3\Delta'(e,f))}$$

Hence, the results can be concluded only with computing the differences of $D_{e0}(G)$, $D_{e3}(G)$ and $D_{e4}(G)$. Then,

$$D_{e0}(G;q) - D_{e3}(G;q) = +\sum_{\{e,f\}\in A_2}\left(q^{(\Delta'(e,f)+1)} - q^{(\Delta'(e,f)+2)}\right) + \sum_{\{e,f\}\in A_4}\left(q^{(3\Delta'(e,f)+1)} - q^{(\Delta'(e,f)+2)}\right)$$
$$+\sum_{\{e,f\}\in A_5}\left(q^{(\Delta'(e,f)+2)} - q^{(\Delta'(e,f)+3)}\right) + \sum_{\{e,f\}\in A_6}\left(q^{(4\Delta'(e,f)+1)} - q^{(3\Delta'(e,f)+1)}\right)$$
$$= (1-q)\sum_{\{e,f\}\in A_2} q^{(\Delta'(e,f)+1)} + \sum_{\{e,f\}\in A_4} q^{(\Delta'(e,f)+2)}\left(q^{(2\Delta'(e,f)-1)}-1\right)$$
$$+(1-q)\sum_{\{e,f\}\in A_5} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_6} q^{(3\Delta'(e,f)+1)}\left(q^{\Delta'(e,f)}-1\right)$$

$$D_{e0}(G;q) - D_{e4}(G;q) = \sum_{\{e,f\}\in A_1}\left(q^{(\Delta'(e,f)+1)} - q^{(\Delta'(e,f)+2)}\right) + \sum_{\{e,f\}\in A_2}\left(q^{(\Delta'(e,f)+1)} - q^{(\Delta'(e,f)+2)}\right)$$
$$+\sum_{\{e,f\}\in A_4}\left(q^{(3\Delta'(e,f)+1)} - q^{(3\Delta'(e,f))}\right) + \sum_{\{e,f\}\in A_5}\left(q^{(\Delta'(e,f)+2)} - q^{(\Delta'(e,f)+3)}\right) + \sum_{\{e,f\}\in A_6}\left(q^{(4\Delta'(e,f)+1)} - q^{(3\Delta'(e,f))}\right)$$
$$= (1-q)\sum_{\{e,f\}\in A_1} q^{(\Delta'(e,f)+1)} + (1-q)\sum_{\{e,f\}\in A_2} q^{(\Delta'(e,f)+1)}$$
$$+(q-1)\sum_{\{e,f\}\in A_4} q^{(3\Delta'(e,f))} + (1-q)\sum_{\{e,f\}\in A_5} q^{(\Delta'(e,f)+2)} + \sum_{\{e,f\}\in A_6} q^{(3\Delta'(e,f))}\left(q^{(\Delta'(e,f)+1)}-1\right)$$

In following, we simplify the new formulates or edge detour indices and edge detour index polynomials for molecular graph of zigzag polyhex nanotubes. We use the notationa p and q for the number of hexagons between two rows and number of rows, respectively. In Fig. 7, you can see the molecular graph of zigzag polyhex nanotubes.

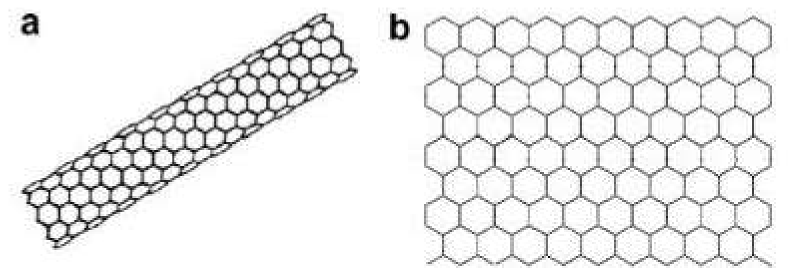

Fig. 7: (a) A zigzag polyhex nanotube, (b) Its 2-dimentional lattice, p = 10 and q = 9

Theorem 3-5: The relations between edge detour indices of the molecular graph G of zigzag polyhex nanotubes are:

$$D_{e0}(G) - D_{e3}(G) = -\left(2p\binom{q}{2} + (q-1)\binom{p}{2}\right)$$

and

$$D_{e0}(G) - D_{e4}(G) = -\binom{3pq-p}{2}$$

Proof: Let G be the molecular graph G of zigzag polyhex nanotubes. Due to the Fig. 7, the subsets A_3, A_4, A_5 and A_6 are empty and then,

$$\binom{|E(G)|}{2} = |A_1| + |A_2|$$

Therefore we have

$$D_{e0}(G) - D_{e3}(G) = -|A_2|$$

and

$$D_{e0}(G) - D_{e4}(G) = -(|A_1| + |A_2|)$$

And since |E(G)| = 3pg-p and

$$|A_2| = 2p\binom{q}{2} + (q-1)\binom{p}{2}$$

we can conclude the desire results.

Theorem 3-6: The relations between edge-detour index polynomials of the molecular graph G of zigzag polyhex nanotubes are:

$$D_{e0}(G;q) - D_{e3}(G;q) = (1-q)\sum_{\{e,f\}\in A_2} q^{(\Delta'(e,f)+1)}$$

and

$$D_{e0}(G;q) - D_{e4}(G;q) = (1-q)\sum_{\{e,f\}\subseteq E(G)} q^{(\Delta'(e,f)+1)}$$

Proof: Let G be the molecular graph G of zigzag polyhex nanotubes. Since the subsets A_3, A_4, A_5 and A_6 are empty and

$$\binom{|E(G)|}{2} = |A_1| + |A_2|$$

the desire results can be concluded easily from Theorem (3-4).

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REFERENCES

1. Harary, F., 1990. Graph theory. Addison-Wesley, Reading, Massachusetts.
2. Buckely, F. and F. Harary, 1990. Distance in graphs. Addison-Wesley, Reading MA.
3. Amic, D. and N. Trinajstic, 1995. On the detour matrix. Croat. Chem. Acta, 68: 53-62.
4. Ivanciuc, O. and A.T. Balaban, 1994. Design of topological indices. Part. 8. Path matrices and derived molecular graph invariants. MATCH Commun. Math. Comput. Chem., 30: 141-152.
5. John, P.E., 1995. Ueber die Berechnung des Wiener index fuer ausgewaehlte Delta-dimensionale Gitterstrukturen. MATCH Commun. Math. Comput. Chem., 32: 207-219.
6. Lukovits, I.., 1996. The detour index. Croat. Chem. Acta, 69: 873-882.
7. Nikolic, S., N. Trinajstic, A.. Juric and Z. Mihalic, 1996. The detour matrix and the detour index of weighted graphs. Croat. Chem. Acta, 69: 1577-1591.
8. Jalal Shahkoohi, R., O. Khormali and A. Mahmiani, The polynomial of detour index for a graph. Submitted.
9. Mahmiani, A., O. Khormali and A. Iranmanesh, 2009. The edge versions of detour index. MATCH Commun. Math. Comput. Chem., 62 (2): 419-431.