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Exact Travelling Wave Solutions for the Generalized Shallow Water Wave (GSWW) Equation

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Abstract: In this paper, the $(\frac{G'}{G})$ -expansion method is employed to obtain exact traveling wave solutions of the generalized shallow water wave (GSWW) equation in forms of the hyperbolic functions and the trigonometric functions. The solutions gained from the proposed method have been verified with those obtained by the Hirota's method and the tanh–coth method. It is shown that the $(\frac{G'}{G})$ -expansion method provides a very effective and powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

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INTRODUCTION

Considered as a fascinating element of nature, nonlinearity is regarded by many scholars as the most significant frontier for the fundamental understanding of nature. Many complex physical phenomena are frequently described and modeled by nonlinear evolution equations (NLEEs), accordingly, the exact or analytical solutions of the discussed nonlinear evolution equations prove to be of utmost importance, which is considered not only a valuable tool in checking the accuracy of computational dynamics, but also a conspicuous help to readily understand the essentials of complex physical phenomena such as the collision of two solitary solutions. In the numerical methods [1, 2], stability and convergence should be considered, so as to avoid divergent or inappropriate results. However, in recent years, a variety of effective analytical and semi-analytical approaches have been suggested to obtain explicit travelling and solitary wave solutions of NLEEs, such as the variational iteration method (VIM) [3-6], (HPM) [7-9], the parameter-expansion method [10], the sine-cosine method [11], the tanh method [12,13], the homotopy analysis method (HAM) [14], the homogeneous balance method [15], the inverse scattering method [16], the Exp-function method [17-26] and others.

Recently, the $(\frac{G'}{G})$ -expansion method, first introduced by Wang *et al*. [29], has become widely used to search for various exact solutions of NLEEs [21, 30-33]. The value of the $(\frac{G'}{G})$ -expansion method is that one treats nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for travelling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation.

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Clarkson and Mansfield [28], investigated the generalized short water wave (GSWW) equation given by

$$u_t - u_{xxt} - \alpha\, u u_t - \beta\, u_x \int^x u_t dx + u_x = 0. \quad (1)$$

Where α and β are non-zero constants.

Many authors have studied some types of solutions of the above Equation. To mention, Wazwaz [27], successfully examined solitary wave solutions to the GSWW equation by means of the Hirota's method, tanh–coth method and Exp-function method.

Considering all the indispensably significant issues mentioned above, the objective of this paper is to investigate the travelling wave solutions of Eq. (1) systematically, by applying the $(\frac{G'}{G})$ -expansion method. Some previously known solutions are recovered as well and, simultaneously, somemore general ones are also proposed.

The $(\frac{G'}{G})$ -expansion Method: Suppose we have a nonlinear partial differential equation (NLEE) for u(x,t) in the form

$$P(u,u_x u_t,u_{xx},u_{tt},u_{xt},...) = 0 \quad (2)$$

Where u(x,t) unkown function and dependent to x, t varabiles and P is a polynomial in u(x,t) and its partial derivatives, in which the highest order derivatives and non-linear terms are involved. The transformation u(x,t)=U(ξ), ξ=x-ωt reduces Eq. (2) to the

$$Q(U, U', U'', \ldots) = 0 \quad (3)$$

Where U=U(ξ) and prime denotes the derivative with respect to ξ and ω are constants. To keep the solution process as simple as possible, the function Q should not be a total ω-derivative of another function. Otherwise, taking integration with respect to ξ further reduces the transformed equation.

Suppose that the solution of Eq. (3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U = \sum_{i=1}^{m} \alpha_i (\frac{G'}{G})^i + \alpha_0, \qquad \alpha_m \neq 0, \quad (4)$$

Where G=G(ξ) satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \quad (5)$$

Where $G' = \frac{dG}{d\xi}$, $G'' = \frac{d^2G}{d\xi^2}$, $\alpha_m \neq 0,\ldots,\alpha_1,\alpha_0,\lambda$ and μ are real constants to be determined later.

So, a direct computation with use from Eqs. (4(and (5) gives

$$U' = -\sum_{i=1}^{m} i\alpha_i [(\frac{G'}{G})^{i+1} + \lambda(\frac{G'}{G})^i + \mu(\frac{G'}{G})^{i-1}], \quad (6)$$

$$U'' = \sum_{i=1}^{m} i\alpha_i [(i+1)(\frac{G'}{G})^{i+2} + (2i+1)\lambda(\frac{G'}{G})^{i+1} + i(\lambda^2 + 2\mu)(\frac{G'}{G})^i + (2i-1)\lambda\mu(\frac{G'}{G})^{i-1} + (i-1)\mu^2(\frac{G'}{G})^{i-2}], \quad (7)$$

and so on, in other hands with using the general silutions of Eq. (5) we have

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0 \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0 \end{cases} \quad (8)$$

To determine U explicitly, we take the following four steps:

Step 1: Determine the integer m by substituting Eq. (4) along with Eq. (5) into Eq. (3) and balancing the highest order nonlinear term(s) and the highest order partial derivative.

Step 2: By substituting Eqs. (4) and (5) into Eq. (3) with the value of m obtained in Step 1 and collecting all term(s) with the same order of $(\frac{G'}{G})$ together, the left-hand side of Eq. (3) converted into polynomial in $(\frac{G'}{G})$. Then setting each coefficient to zero, we obtained a set of algeratic equations for λ, μ, ω, α_0 and α_i.

Step 3: Solve the system of algebratic equations obtained in step 2 for ω, α_0 and α_i by use of Maple.

Step 4: By substituting the results obtained in the above steps, we can obtain a series of fundamental solutions of Eq. (3).

The Generalized Shallow Water Wave (GSWW): In this section, we investigate the generalized shallow water wave (GSWW) with the $(\frac{G'}{G})$ -expansion method to construct the exact traveling wave solutions.

We consider the generalized shallow water wave (GSWW).

$$u_t - u_{xxt} - \alpha\, uu_t - \beta\, u_x \int^x u_t dx + u_x = 0. \tag{9}$$

Making the transformation u=$u = v_x$, Eq (9) becomes

$$v_{xt} - v_{xxxt} - \alpha\, v_x v_{xt} - \beta v_{xx}\, v_t v_{xx} = 0. \tag{10}$$

Using the wave variable ξ=x-ωt, the system (10), is carrried to a system of ODEs

$$(1-\omega)v'' + \omega(\alpha+\beta)v'v'' + \omega\, v'''' = 0, \tag{11}$$

and integrating Eq (11), once with respect to ξ and setting the integration constant as zero yields

$$(1-\omega)v' + \frac{\omega}{2}(\alpha+\beta)(v')^2 + \omega\, v''' = 0, \tag{12}$$

Balancing $(v')^2$ with v''' in Eq (12), gives 2m+2=m+3 so that m=1.

Suppose that the solution of ODE (12) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$v = \alpha_1(\frac{G'}{G}) + \alpha_0, \qquad \alpha_1 \neq 0, \tag{13}$$

Where α_0, α_1 are unknown constants that to be determined later.

On substituting (13) into (12), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting each coefficient to zero, we obtain the following system of algebraic equations for λ, μ, ω, α_0 and α_1, as follows:

$$(\frac{G'}{G})^0: \quad \frac{1}{2}\omega\alpha_1^2\alpha\mu^2 + \frac{1}{2}\omega\alpha_1^2\beta\mu^2 - \omega\alpha_1\lambda^2\mu - \alpha_1\mu + \alpha_1\omega\mu - 2\omega\alpha_1\mu^2,$$

$$(\frac{G'}{G})^1: \quad \omega\alpha_1\lambda - \omega\alpha_1\lambda^3 - \alpha_1\lambda + \omega\alpha_1^2\alpha\lambda\mu + \omega\alpha_1^2\beta\lambda\mu - 8\omega\alpha_1\lambda\mu,$$

$$(\frac{G'}{G})^2: \quad -8\alpha_1\omega\mu - 7\omega\alpha_1\lambda^2 - \alpha_1 + \omega\alpha_1 + \omega\alpha_1^2\alpha\mu + \frac{1}{2}\omega\alpha_1^2\alpha\lambda^2 + \omega\alpha_1^2\beta\mu + \frac{1}{2}\omega\alpha_1^2\beta\lambda^2,$$

$$(\frac{G'}{G})^3: \quad -12\omega\alpha_1\lambda + \omega\alpha_1^2\alpha\lambda + \omega\alpha_1^2\beta\mu,$$

$$(\frac{G'}{G})^4: \quad \frac{1}{2}\omega\alpha_1^2\alpha + \frac{1}{2}\omega\alpha_1^2\beta - 6\omega\alpha_1.$$

On solving the above algebraic equations by using the Maple, we get

$$\alpha_1 = \frac{12}{\alpha+\beta}, \qquad \alpha_0 = \alpha_0, \qquad \omega = \frac{1}{\lambda^2 - 4\mu - 1}, \tag{14}$$

Where λ, μ, α_0 are arbitrary constants and α, β nonzero constants.
Therefore, by substitute (14) into (13), we can obtain that

$$v = \frac{12}{\alpha+\beta}\left(\frac{G'}{G}\right) + \alpha_0, \tag{15}$$

Substituting the general solutions (8) into Eq. (15), we have two types of travelling wave solutions of the generalized shallow water wave (GSWW).

When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solutions,

$$v(\xi) = \frac{12}{\alpha+\beta}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right) - \alpha_0, \tag{16}$$

in which ξ=x-ωt and C_1,C_2 are arbitrary constants.
When $\lambda^2 - 4\mu < 0$, we obtain trigonometric function solutions,

$$v(\xi) = \frac{12}{\alpha+\beta}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right) - \alpha_0, \tag{17}$$

Where ξ=x-ωt and C_1,C_2 are arbitrary parameters that can be determined by the related initial and boundary conditions.
Now, to obtain some special cases of the general solution (16), we set $C_1 \neq 0$, $C_1 = 0$ and $C_1 = 0$, $C_2 \neq 0$ respectively, then it is obvious that

$$v_1(x,t) = \frac{6}{\alpha+\beta}\sqrt{\frac{\omega-1}{\omega}}\tanh\left[\frac{1}{2}\sqrt{\frac{\omega-1}{\omega}}(x-\omega t)\right] - \frac{6\lambda}{\alpha+\beta} + \alpha_0, \tag{18}$$

$$v_2(x,t) = \frac{6}{\alpha+\beta}\sqrt{\frac{\omega-1}{\omega}}\coth\left[\frac{1}{2}\sqrt{\frac{\omega-1}{\omega}}(x-\omega t)\right] - \frac{6\lambda}{\alpha+\beta} + \alpha_0, \tag{19}$$

in which $\omega > 1$.
Recall that u(x,t)=v_x(x,t), we get the formal solitary wave solution of Eq. (1) as follows:

$$u_1(x,t) = \frac{3(\omega-1)}{\omega(\alpha+\beta)}\sec h^2\left[\frac{1}{2}\sqrt{\frac{\omega-1}{\omega}}(x-\omega t)\right], \tag{20}$$

$$u_2(x,t) = \frac{3(\omega-1)}{\omega(\alpha+\beta)}\coth^2\left[\frac{1}{2}\sqrt{\frac{\omega-1}{\omega}}(x-\omega t)\right], \tag{21}$$

valid for $\omega > 1$ follow immediately.

If we choose $C_1 \neq 0$, $C_2 = 0$ and $C_1 = 0$, $C_2 \neq 0$, in Eq. (17), respectively, then the general solution (17) reduces to

$$v_3(x,t) = -\frac{6}{\alpha+\beta}\sqrt{\frac{1-\omega}{\omega}}\tan[\frac{1}{2}\sqrt{\frac{1-\omega}{\omega}}(x-\omega t)] - \frac{6\lambda}{\alpha+\beta} + \alpha_0, \quad (22)$$

$$v_4(x,t) = -\frac{6}{\alpha+\beta}\sqrt{\frac{1-\omega}{\omega}}\cot[\frac{1}{2}\sqrt{\frac{1-\omega}{\omega}}(x-\omega t)] - \frac{6\lambda}{\alpha+\beta} + \alpha_0, \quad (23)$$

Recall that u(x,t)=v_x(x,t), Then we can obtain the general trigonometric function solutions of Eq. (1) as follows

$$u_3(x,t) = \frac{3(\omega-1)}{\omega(\alpha+\beta)}\sec^2[\frac{1}{2}\sqrt{\frac{1-\omega}{\omega}}(x-\omega t)], \quad (24)$$

$$u_4(x,t) = \frac{3(\omega-1)}{\omega(\alpha+\beta)}\csc^2[\frac{1}{2}\sqrt{\frac{1-\omega}{\omega}}(x-\omega t)], \quad (25)$$

Where ω<1.

Comparing the particular cases of our general solutions, Eqs. (20, 21, 24, 25), with Wazwaz's results, Eqs. (57-60) in [27], it can be seen that the results are the same.

Remark 1: All the travelling wave solutions of Eq. (1) obtained by the tanh–coth method and the Hirota's method in [27] are particular cases of our general solutions.

Remark 2: We have verified all the obtained solutions by putting them back into the original equation (1) with the aid of Maple 13.

CONCLUSIONS

To sum up, the purpose of the study is to show that exact travelling wave solutions of the GSWW equation can be obtained by the $(\frac{G'}{G})$ -expansion method. These solutions include hyperbolic function solutions and trigonometric function solutions. When the parameters are taken as special values, the solitary wave solutions are derived from the hyperbolic function solutions. The final results from the proposed method have been compared and verified with those obtained by the Hirota's method and the tanh–coth method. We also found more general solutions which are not obtained by the other existed methods. Overall, the results reveal that the $(\frac{G'}{G})$ -expansion method is a powerful mathematical tool to solve nonlinear partial differential equations (NPDEs) in the terms of accuracy and efficiency. This is important, since systems of NPDEs have many applications in engineering.

REFFERENCES

1. Borhanifar, A. and R. Abazari, 2010. Numerical study of nonlinear Schrodinger and coupled Schr¨odinger equations by differential transformation method. Optics Communications, 283: 2031-2036.
2. Borhanifar, A., M.M. Kabir and A. Hossein Pour, 2011. A Numerical Method for Solution of the Heat Equation with Nonlocal Nonlinear Condition. World Applied Sciences J., 13(11): 2405-2409.
3. He, J.H., G.C. Wu and F. Austin, 2010. The variational 16. Ablowitz, M.J. and H. Segur, 1981. Solitons and iteration method which should be followed. Nonlinear inverse scattering transform. SIAM, Philadelphia. Science Letters A, 1(1): 1-30.
4. He, J.H., 2007. Variational iteration method-Some recent results and new interpretations. J. Computational and Applied Mathematics, 207: 3-17.
5. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Variational Iteration Method for Solving Flierl-petviashivili Equation Using He's Polynomials and Pade´ Approximants. World Applied Sciences J., 6(9): 1298-1303.
6. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Modified Variational Iteration Method for Solving Sine-Gordon Equations. World Applied Sciences J., 6(7): 999-1004.
7. He, J.H., 2006. New interpretation of homotopy perturbation method. Int. J. Mod. Phys. B, 20(18): 2561-8.
8. Yildirim, A., S. Sariaydin and S.T. Mohyud-Din, 2010. Homotopy Perturbation Method for Boundary Layer Flow on a Continuous Stretching Surface. Nonlinear Science Letters A, 1(4): 385-390.

9. Ali Khan, R. and M. Usman, 2010. Long Time Dynamics of Forced Oscillations of the Korteweg-de Vries Equation Using Homotopy Perturbation Method. Studies in Nonlinear Sciences, 1(3): 57-62.
10. Wang, S.Q. and J.H. He, 2008. Nonlinear oscillator with discontinuity by parameter-expansion method. Chaos, Solitons and Fractals, 35(4): 688-691.
11. Wazwaz, A.M., 2005. A class of nonlinear fourth order variant of a generalized Camassa-Holm equation with compact and noncompat solutions. Appl. Math. Comput., 65(2): 485-501.
12. Wazwaz, A.M., 2005. The Camassa-Holm-KP equations with compact and noncompact traveling wave solutions. Applied Mathematics and Computation, 170: 347-360.
13. Malfliet, W., 2005. The tanh method: a tool for solving certain classes of non-linear PDEs. Mathematical Methods in the Applied Sciences, 28(17): 2031-2035.
14. Hosseini, M.M., S.T., Mohyud-Din, S.M. Hosseini and M. Heydari, 2010. Study on Hyperbolic Telegraph Equations by Using Homotopy Analysis Method. Studies in Nonlinear Sciences, 1(2): 50-56.
15. Fan, E. and H. Zhang, 1998. A note on the homogeneous balance method. Phys. Lett. A, 246: 403-406.
16. Ablowitz, M.J. and H. Segur, 1981. Solitons and inverse scattering transform. SIAM, Philadelphia.
17. He, J.H. and X.H. Wu, 2006. Exp-function method for nonlinear wave equations. Chaos, Solitons and Fractals, 30(3): 700-8.
18. Wu, X.H. and J.H. He, 2007. Solitary solutions, periodic solutions and compaction-like solutions using the Exp-function method. Computers and Mathematics with Applications, 54: 966-986.
19. Zhang, S., 2010. Exp-function Method: Solitary, Periodic and Rational Wave Solutions of Nonlinear Evolution Equations. Nonlinear Science Letters A, 1(2): 143-146.
20. Khajeh, A., A. Yousefi-Koma, M. Vahdat and M.M. Kabir, 2010. Exact Travelling Wave Solutions for Some Nonlinear Equations Arising in Biology and Engineering. World Applied Sciences J., 9(12): 1433-1442.
21. Kabir, M.M., 2011. Analytic solutions for generalized forms of the nonlinear heat conduction equation. Nonlinear Analysis: Real World Applications, 12: 2681-2691.
22. Kabir, M.M. and A. Khajeh, 2009. New explicit solutions for the Vakhnenko and a generalized form of the nonlinear heat conduction equations via Exp-function method. Int. J. Nonlinear Sciences and Numerical Simulation, 10(10): 1307-1318.
23. Kabir, M.M., A. Khajeh, E. Abdi Aghdam and A. Yousefi Koma, 2011. Modified Kudryashov method for finding exact solitary wave solutions of higher-order nonlinear equations. Math. Methods Appl. Sci., 34: 213-219.
24. Borhanifar, A. and M.M. Kabir, 2009. New periodic and soliton solutions by application of Exp-function method for nonlinear evolution equations. J. Computational and Applied Mathematics, 229: 158-167.
25. Borhanifar, A., M.M. Kabir and M. Vahdat Lasemi, 2009. New periodic and soliton wave solutions for the generalized Zakharov system and (2+1)-dimensional Nizhnik-Novikov-Veselov system. Chaos, Solitons and Fractals, 42: 1646-1654.
26. Borhanifar, A. and M.M. Kabir, 2010. Soliton and Periodic solutions for (3+1)-dimensional nonlinear evolution equations by Exp-function method. Applications and Applied Mathematics: International Journal (AAM), 5(1): 59-69.
27. Wazwaz, A.M., 2008. Solitary wave solutions of the generalized shallow water wave (GSWW) equation by Hirota's method, tanh–coth method and Exp-function method. Applied Mathematics and Computation, 202: 275-286.
28. Clarkson, P.A. and E.L. Mansfield, 1994. On a shallow water wave equation. Nonlinearity, 7: 975-1000.
29. Wang, M., X. Li and J. Zhang, 2008. The $(\frac{G'}{G})$-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics. Phys. Lett. A, 372: 417-423.
30. Zedan, H.A., 2010. New classes of solutions for a system of partial differential equations by $(\frac{G'}{G})$-expansion method. Nonlinear Science Letters A, 1(3): 219-238.
31. Abazari, R., 2010. The $(\frac{G'}{G})$-expansion method for Tzitzéica type nonlinear evolution equations. Mathematical and Computer Modelling, 52: 1834-1845.
32. Kabir, M.M., A. Borhanifar and R. Abazari, 2011. Application of $(\frac{G'}{G})$-expansion method to Regularized Long Wave (RLW) equation. Computers and Mathematics with Applications, 61(8): 2044-2047.
33. Kabir, M.M. and R. Bagherzadeh, 2011. Application of $(\frac{G'}{G})$-expansion method to Nonlinear Variants of the (2+1)-Dimensional Camassa-Holm-KP Equation. Middle-East J. Scientific Research, 9(5): 602-610.