Middle-East Journal of Scientific Research 10 (2): 191-195, 2011
ISSN 1990-9233


# Solution of Fifth Order Boundary Value Problems in Reproducing Kernel Space

*Ghazala Akram and Hamood ur Rehman*

Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan

**Abstract:** The approximate solutions to the fifth-order boundary-value problems are solved using the reproducing kernel space method. The argument is based on the reproducing kernel space . The numerical procedure is applied on linear and nonlinear problems. The approach provides the solution in the form of a convergent series with easily computable components. Two numerical examples are given to illustrate the implementation and degree of efficiency of the method. The method introduced compared with those developed by [1-4], reveals that the present method is more effective and convenient L.

**Key words:** Gram-Schmidt orthogonal process · Reproducing kernel · Searching least value (SLV) method

## INTRODUCTION

Little description of the solution of fifth order boundary value problems appears in the numerical analysis literature. These problems generally arise in the mathematical modelling of viscoelastic flows and other branches of mathematical, physical and engineering sciences [5-7]. The conditions for the existence and uniqueness of the solution of such problems can be found in [8]. Siddiqi and Ghazala [3] presented nonpolynomial spline method for the numerical solution of the fifth-order linear special case boundary value problems. Siddiqi and Ghazala [4] used polynomial sextic spline method for the solution of linear fifth-order boundary value problems and the method is observed to be second-order convergent.

El-Gamel [1] presented Sinc-Galerkin method for the solution of fifth-order boundary value problems with two-point boundary conditions. Shen [2] solved fifth-order boundary value problems using the homotopy perturbation method. A reproducing kernel Hilbert space is a useful framework for constructing approximate solutions of differential equations [9-11]. In this paper a reproducing kernel method is used for the solution of fifth order boundary value problem.

Consider the following fifth order two-point boundary value problem (BVP).

$$\left.\begin{aligned} &u^{(5)}(x)+\sum_{i=0}^{4} a_i(x)u^{(i)}(x)=f(x,u(x)), \quad 0<x\le 1\\ &u(0)=A_0,\ u(1)=A_1,\ u^{(1)}(0)=A_2,\\ &u^{(1)}(1)=A_3,\ u^{(2)}(0)=A_4, \end{aligned}\right\} \tag{1.1}$$

Where $A_i(x)$ $i$ = 0,1,2,3,4 are finite real constants and the $f(x,u(x))$, $a_i(x)$, i=0,1,2,3,4 are continuous on [0,1]. Let $L = d^5 / dx^5$ be the differential operator and homogenization of the boundary conditions of system (1.1) can be transformed into the following form

$$\left.\begin{aligned} &Lu=f(x,u(x)), \qquad 0<x\le 1\\ &u(0)=0,\ u(1)=0,\ u^{(1)}(0)=0,\\ &u^{(1)}(1)=0,\ u^{(2)}(0)=0, \end{aligned}\right\} \tag{1.2}$$

Thus, the solution of system (1.2) provides the solution of the system (1.1).

The rest of this paper is organized as under:

In Section 2, the reproducing kernel spaces and the reproducing kernel function are given. The approximate solution of problems (1.2) with boundary conditions is presented in Section 3. The numerical examples are presented to demonstrate the accuracy of the method in Section 4.

**Corresponding Author:** Ghazala Akram, Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan.

**Reproducing Kernel Spaces:**

- Reproducing kernel space $W_2^6[0, 1]$ is defined by $W_2^6[0, 1] = \{u(x)|u^{(i)}, i = 0,1,...,5$ are absolutely continuous real valued functions in $[0,1]$, $u^{(6)} \in L^2[0,1]\}$.

The inner product and norm in $W_2^6[0, 1]$ are defined by

$$<u(y),v(y)>=\sum_{i=0}^{2}u^{(i)}(0)v^{(i)}(0)+\sum_{i=0}^{2}u^{(i)}(1)v^{(i)}(1)+\int_0^1 u^{(6)}(y)v^{(6)}(y)dy \quad (2.1)$$

$$\|u\|=\sqrt{<u,u>_{W_2^6}}, \quad u(x),v(x)\in W_2^6[0,1] \quad (2.2)$$

- Reproducing kernel space $W_2^1[0, 1]$ is defined by $W_2^1[0, 1] = \{u(x)|u$ is absolutely continuous real valued functions in $[0,1]$, $u, u^{(1)} \in L^2[0, 1]\}$. . The inner product and norm in $W_2^1[0, 1]$ are defined as

$$<u(y),v(y)>_{W_2^1}=u(0)v(0)+\int_0^1 u^{(1)}(y)v^{(1)}(y)dy$$

$$\|u\|=\sqrt{<u,u>_{W_2^1}}, \quad u(x),v(x)\in W_2^1[0,1] \quad (2.3)$$

Lin and Cui [12], proved that $W_2^1[0, 1]$ is reproducing kernel Hilbert space and its reproducing kernel is given by

$$Q_x(y)=\frac{1}{2\sinh 1}[\cos(x+y-1)+\cosh(|x-y|-1)]$$

**Theorem 2.1:** The space $W_2^6[0, 1]$ is a reproducing kernel Hilbert space. i.e $\forall\, u(y) \in W_2^6[0, 1]$ and each fixed $x \in [0, 1]$, $y \in [0, 1]$ there exists $k(x, y) \in W_2^6[0, 1]$ such that

$$<u(y), k(x, y)>=u(x)$$

and $k(x, y)$ is called the reproducing kernel function of space $W_2^6[0, 1]$ The reproducing kernel function $k(x, y)$ is given by

$$k(x,y)=\begin{cases} h(x,y) & y\le x, \\ h(y,x) & y\le x. \end{cases}$$

Where $h(x, y) =$

$$(9979320x^3-19958745x^4+9979481x^5-165x^8+165x^9-66x^{10}+10x^{11})y^3/39916800+$$
$$(-19958745x^3+39917840x^4-19959333x^5+330x^7-165x^9+88x10-15x11)y^4/39916800+$$
$$(9979481x^3-19959333x^4+9980286x^5-462x^6+55x^9-33x^{10}+6x^{11})y^5/39916800+(-x^2+$$
$$3x^3-3x^4+x^5)y^9/725760+(x-6x^3+8x^4-3x^5)y^{10}/3628800+(-1+10x^3-15x^4+$$
$$6x^5)y^{11}/39916800$$

**The Exact and Approximate Solution:** The solution of Eq. (1. 2) is given in the reproducing kernel Hilbert space $W_2^6[0, 1]$ and the linear operator $L : W_2^6[0, 1] \to W_2^1[0, 1]$ is bounded. Choose a countable dense subset $D=\{x_i\}_{i=1}^{\infty}$ in the domain [0, 1], and let

$$\varphi_i(y)=Q_{x_i}(y), \quad i\in N \quad (3.1)$$

Where $Q_{x_i}(y)\in W_2^1[0,1]$ is reproducing kernel of $W_2^1[0, 1]$. Further assume that $\psi_i(x) = (L^*\varphi_i)(x)$, where L* is the conjugate operator of L.

**Theorem 3.1:** $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete syste of $W_2^6[0, 1]$ and $\psi_i(x)=L_y k(x,y)\big|_{y=x_i}$.

**Proof:** For each fixed $u(x)\in W_2^6[0,1]$, $\langle u(x),\psi_i(x)\rangle=0 (i=1,2,...)$, which implies

$$\left\langle u(x),(L^*\varphi_i)(x)\right\rangle=<(Lu)(x),Q_{x_i}(x)>=(Lu)(x_i)=0 \quad (3.2)$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in [0, 1], (Lu)(x) = 0, which implies u = 0 from the $L^{-1}$. From Eq. (2. 4), it can be written as

$$\psi_i(x) = <\psi_i(y),(k(x,y)> = <(L^*\varphi_i(x)),(k(x,y)> = <\varphi_i(y),L(k(x,y)> = L_y k(x,y)|_{y=x_i}$$

To orthonormalize the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$ in the reproducing kernel space $W_2^6[0, 1]$. Gram-Schmidt process can be used as

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad i = 1,2,3,... \tag{3.3}$$

**Theorem 3.2:** If $\{x_i\}_{i=1}^{\infty}$ is dense in [0, 1] and the solution of Eq. (1. 2) is unique, for all $u(x) \in W_2^6[0, 1]$, the series is convergent in the norm $\|\cdot\|_{W_2^6}$. If u(x) is exact solution then the solution of system (1. 2) has the form

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\alpha_k\overline{\psi}_i(x)$$

Where $\alpha_k = f(x_k,u(x_k)),(k=1,2,...), \{x_i\}_{i=1}^{\infty}$ is dense in [0,1].

**Proof:** Since $u(x) \in W_2^6[0, 1]$ and can be expanded in the form of Fourier series about normal orthogonal system $\{\psi_i(x)\}_{i=1}^{\infty}$ as

$$u(x) = \sum_{i=1}^{\infty} <u(x),\overline{\psi}_i(x)>\overline{\psi}_i(x) \tag{3.4}$$

Since the space $W_2^6[0, 1]$ is Hilbert space so the series $\sum_{i=1}^{\infty} <u(x),\overline{\psi}_i(x)>\overline{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2^6}$.

From Eqns. (3. 3) and (3. 4), it can be written as

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty} <u(x),\overline{\psi}_i(x)>\overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} <u(x),\psi_i(x)>\overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} <u(x),L^*\varphi_k(x)>\overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} <Lu(x),\varphi_k(x)>\overline{\psi}_i(x)
\end{aligned}$$

If u(x) is the exact solution of Eq. ( 1. 2 ) and Lu = f (x,u(x)), the

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} <f(x,u(x)),\varphi_k(x)>\overline{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x,u(x))\overline{\psi}_i(x).
\end{aligned}$$

The approximate solution of u(x) is given by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k,u(x_x))\overline{\psi}_i(x) \tag{3.5}$$

Where

$\alpha_k = f(x_k,u(x_k))$, $k = 1,2,...$

**Remark**: If system ( 1. 2 ) is linear, i.e. f(x; u(x)) = f(x), then then solution can be obtained directly from Eq. (3.5).

If system (1. 2) is nonlinear the approximate solution can be obtained using the following method. In order to obtain in (3. 5), substitute

$$\sum_{k=1}^{\infty}[f(x_k,u_n(x_k)) - \alpha_k]^2 = 0.$$

$$J(\alpha_1,\alpha_2,...,\alpha_k) = \sum_{k=1}^{\infty}[f(x_k,u_n(x_k)) - \alpha_k]^2,$$

if $\alpha_1$, $\alpha_2$,..., $\alpha_n$ are least value points then the approximate solutions $u_n$ (x) are obtained. Next we will give the following algorithm to obtain $u_n$ (x) in Eq. (3.5).

**Step 1:** Take initial values $\alpha_k^0$, $k = 1,2,...,i$

**Step 2:** Substitute $\alpha_k^0$, $k = 1,2,...,i$ in Eq. (3.5) and obtain $u_n^0(x)$

**Step 3:** Calculate $J(\alpha_1^0,\alpha_2^0,...,\alpha_n^0)$

**Step 4:** If $J(\alpha_1^0,\alpha_2^0,...,\alpha_n^0) < 10^{-30}$ then the calculation process terminates; otherwise, substitute $u_n^0(x)$ into Eq. (3.5) then $\alpha_k^1$, $k = 1,2,...,i$ is yielded and go to next step.

**Step 5:** Calculate $J(\alpha_1^1,\alpha_2^1,...,\alpha_n^1)$ if $J(\alpha_1^1,\alpha_2^1,...,\alpha_n^1) < J(\alpha_1^0,\alpha_2^0,...,\alpha_n^0)$ then replace $\alpha_1^0$ by $\alpha_1^1$ and return to step 2, otherwise will be used.

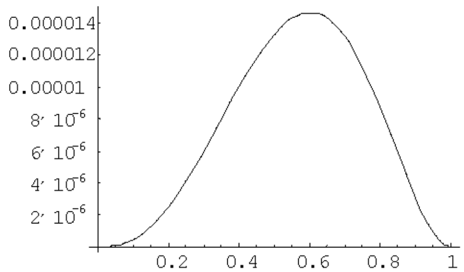


Fig. 1: $|u-u_{10}|$

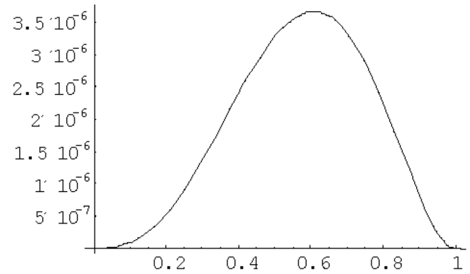


Fig. 2: $|u-u_{20}|$

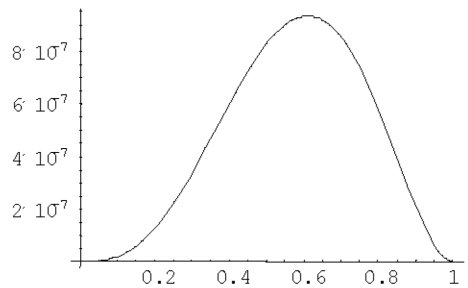


Fig. 3: $|u-u_{40}|$

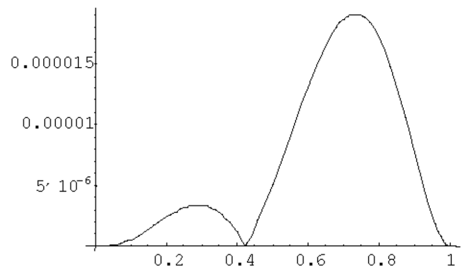


Fig. 4: $|u-u_{10}|$

To illustrate the applicability and effectiveness of our method, three numerical examples are constructed. All the numerical computations performed using Mathematica version 5.2.

**Numerical Examples**

**Example 4.1:** The following boundary value problem is considered

$$\left.\begin{array}{ll} u^{(5)}(x)+u(x)=-(15+10x)e^{x}, & 0\le x\le 1 \\ u(0)=0,\ u(1)=0,\ u^{(1)}(0)=1, & \\ u^{(1)}(1)=-e,\ u^{(2)}(0)=0, & \end{array}\right\} \quad (4.1)$$

The exact solution of the Example 4.1 is $u(x) = x(1-x)e^x$. The comparison of the errors in absolute values between the method developed in this paper and that of Siddiqi and Ghazala [3, 4] is shown in Table 1 and Figures 1, 2 and 3.

**Example 4.2:** Consider the nonlinear fifth-order boundary value problem

$$\left.\begin{array}{ll} u^{(5)}(x)+u^{(4)}(x)+e^{-2x}u^{2}(x)=2e^{x}+1, & 0<x<1 \\ u(0)=1,\ u(1)=e,\ u^{(1)}(0)=1, & \\ u^{(1)}(1)=e,\ u^{(2)}(0)=1 & \end{array}\right\} \quad (4.2)$$

The exact solution of the Example 4.2 is $u(x) = e^x$

Table 1: Comparison of the numerical results for problem 4.1

| h | Present method | Siddiqi and Ghazala [3] | Siddiqi and Ghazala [4] |
|---|---|---|---|
| 1/10 | 1.46 E-05 | 1.287 E-04 | 2.2593 E-04 |
| 1/20 | 3.65 E-06 | 2.7918 E-05 | 1.33 E-05 |
| 1/40 | 9.32 E-07 | 9.3983 E-06 | 5.2812 E-06 |

Table 2: Comparison of the numerical results for problem 4.2

| X | Present method n = 10 | Shen [2] | El-Gamel [1] |
|---|---|---|---|
| 0.0 | 0 | 0 | 0 |
| 0.01 | 7.79 E-10 | 1.2 E-9 | 0 |
| 0.1184 | 7.83 E-7 | 7.0 E-6 | 0 |
| 0.1517 | 1.36 E-6 | 1.4 E-5 | 1.0 E-4 |
| 0.2410 | 3.005 E-6 | 4.6 E-4 | 0 |
| 0.3604 | 2.52 E-6 | 1.0 E-4 | 1.0 E-4 |
| 0.4287 | 2.57 E-7 | 1.0 E-4 | 0 |
| 0.5 | 5.04 E-6 | 1.9 E-4 | 2.0 E-4 |
| 0.6395 | 1.58 E-5 | 1.0 E-4 | 1.0 E-4 |
| 0.8482 | 1.33 E-5 | 9.8 E-5 | 2.0 E-4 |
| 0.9996 | 2.10 E-10 | 1.2 E-9 | 2.0 E-4 |
| 1.0 | 0 | 0 | 0 |

The comparison of the errors in absolute values between the method developed in this paper and that of El-Gamel [1] and Shen [2] is shown in.

## CONCLUSION

The reproducing kernel space method is used to find the solution of linear and nonlinear fifth order boundary value problems and the approximate solutions of such problems are given in the form of series. This method avoids the complexity provided by other numerical approaches. The results show that the convergence and accuracy of the method for numerically analyzed fifth order boundary value problems are in a good agreement with the analytical solutions. The very good accuracy of the proposed method has been shown on some linear and nonlinear problems. It is observed that the errors in absolute values are better than compared ones. The numerical results show that only a few number of iteration steps can be used for numerical purpose with a high degree of accuracy. Therefore, the present method is an accurate and reliable analytical technique for eight-order boundary value problems. Mathematica version 5.2 is used for all computational work.

## REFERENCES


1. Mohamed El-Gamel, 2007. Sinc and the numerical solution of fifth-order boundary value problems, Applied Mathematics and Computation, 187: 1417-1433.
2. Shaowei Shen, 2007. Application of homotopy perturbation method to the fifth-order boundary value problems, Int. J. Contemp. Math. Sci., 2(25): 1227-1236.
3. Shahid S. Siddiqi and Ghazala Akram, 2006. Solution of fifth order boundary value problems using nonpolynomial spline technique, Applied Mathematics and Computation, 175: 1574-1581.
4. Sextic Spline Solutions of Fifth Order Boundary Value Problems, 2007. Applied Mathematics Letters 20: 591-597.
5. Davies, A.R., A. Karageoghis and T.N. Phillips, 1988. Spectral glarkien methods for the primary two-point boundary-value problems in modeling viscelastic flows, Internat. Numer. Methods Engrg., 26: 647-662.
6. Karageoghis, A., T.N. Phillips and S.A.R. Davies, 1998. A.r. davies, spectral collocation methods for the primary two-point boundary-value problems in modeling viscelastic flows, Internat. Numer. Methods Engrg., 26: 805-813.
7. Fyfe, D.J., 1975. Linear dependence relations connecting equal interval nth degree splines and their derivatives, Applied Mathematics and Computation, 7: 398-406.
8. Agarwal, R.P., 1986. Boundary value problems for high order differential equations, World Scientific, Singapore.
9. Ghazala, Akram and Hamood Ur Rehman, 2011. Solution of first order singularly perturbed initial value problem in reproducing kernel hilbert space, European J. Scientific Res., 53(4): 516-523.
10. Cui, M.G. and F.Z. Geng, 2007. A computational method for solving one-dimensional variable-coefficient burgers equation, Appl. Math. Comput., 188: 1389-1401.
11. Geng, F.Z. and M.G. Cui, 2007. Solving singular two-point boundary value problem in reproduc- ing kernel space, J. Computational and Appl. Mathematics, 205: 6-15.
12. Chunli, Li and Minggen Cui, 2003. The exact solution for solving a class of nonlinear operator equations in the reproducing kernel space, Appl. Math. Compu., 143(2-3): 393-399.