Computational and Applied Mathematical Sciences 31 (2): 31-32 2018
ISSN 2222-1328
© IDOSI Publications, 2018
DOI: 10.5829/idosi.cams.2018.31.32

On a Combinatorial Identity of Cheon-Seol-Elmikkawy

C.G. León-Vega, J. López-Bonilla and S. Vidal-Beltrán

ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We exhibit that a combinatorial relation obtained by Cheon-Seol-Elmikkawy is a particular case of an expression of Gould.

Key words: Gauss hypergeometric function · Combinatorial identity

INTRODUCTION

Cheon-Seol-Elmikkawy [1] used Riordan arrays to deduce the following property:

$$A \equiv \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k} \equiv B, \tag{1}$$

However, the relation (3.18) in Gould [2] indicates the Ljunggren's identity [3]:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k (y-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k} y^k, \tag{2}$$

Which implies (1) for $y = 1$.

Now we shall employ the Gauss hypergeometric function [4] to give an alternative proof of (1). In fact:

$$A = (1-x)^n \sum_{k=0}^{\infty} t_k, \quad t_k = \frac{(n+k)!}{k!^2 (n-k)!} (\frac{x}{1-x})^k \quad \therefore \quad \frac{t_{k+1}}{t_k} = \frac{(k+n+1)(k-n)}{(k+1)^2} . \frac{x}{x-1}, \tag{3}$$

Hence [5-11]:

$$A = (1-x)^n \ {}_2F_1\left(-n, n+1; 1; \frac{x}{x-1}\right); \tag{4}$$

But we have the Pfaff's identity [4, 12]:

$$(1-x)^{-\alpha} \ {}_2F_1\left(\alpha, \gamma-\beta;\ \gamma; \frac{x}{x-1}\right) = \ {}_2F_1(\alpha, \beta;\ \gamma; x), \tag{5}$$

Then from (4) and (5) for $\alpha = \beta = -n$, $\gamma = 1$ we deduce the result:

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México. E-mail: jlopezb@ipn.mx.

$$A = {}_2F_1(-n,-n;1;x). \tag{6}$$

Similarly:

$$B = x^n \sum_{k=0}^{\infty} p_k, \quad p_k = \binom{n}{k}^2 x^{-k} \quad \therefore \quad \frac{p_k+1}{p_k} = \frac{(k-n)^2}{(k+1)^2}.\frac{1}{x}, \tag{7}$$

Thus [5-11]:

$$B = x^n \; {}_2F_1\left(-n,-n;1;\frac{1}{x}\right). \tag{8}$$

On the other hand, we know the expression [13]:

$$(\beta)_n \, x^n \; {}_2F_1\left(-n,\gamma-\beta;1-n-\beta;\frac{1}{x}\right) = (\gamma-\beta)_n \; {}_2F_1(-n,\beta;1-n+\beta-\gamma;x), \tag{9}$$

Therefore, from (8) and (9) for $\beta = -n$, $\gamma = -2n$ we obtain that:

$$B = {}_2F_1(-n,-n;1;x); \tag{10}$$

Finally, the results (6) and (10) imply (1), q.e.d.

REFERENCES

1. Gi-Sang Cheon, Han-Guk Seol and M. Elmikkawy, 2006. New identities for Stirling numbers via Riordan arrays, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 13(4): 311-318.
2. Gould, H.W., 1972. Combinatorial identities, Morgantown, W. Va.
3. Ljunggren, W., 1947. Et elementaert for en formel av A. C. Dixon, Norsk Mat. Tidssrift, 29: 35-38.
4. Andrews, G.E., R. Askey and R. Roy, 2000. Special functions, Cambridge University Press.
5. Petkovsek, M., H.S. Wilf and D. Zeilberger, 1996. A = B, symbolic summation algorithms, A. K. Peters, Wellesley, Mass.
6. Koepf, W., 1998. Hypergeometric summation, Vieweg, Braunschweig / Wiesbaden.
7. Koepf, W., 2003. Computer algebra algorithms for orthogonal polynomials and special functions, in ‘Orthogonal polynomials and special functions’, Eds. E. Koelink, W. van Assche; Springer-Verlag, Berlin, pp: 1-24.
8. Hannah, J.P., 2013. Identities for the gamma and hypergeometric functions: an overview from Euler to the present, Master of Science Thesis, University of the Witwatersrand, Johannesburg, South Africa.
9. Guerrero-Moreno, I. and J. López-Bonilla, 2016. Combinatorial identities from the Lanczos approximation for gamma function, Comput. Appl. Math. Sci., 1(2): 23-24.
10. López-Bonilla, J., R. López-Vázquez and S. Vidal-Beltrán, 2018. Hypergeometric approach to the Munarini and Ljunggren binomial identities, Comput. Appl. Math. Sci., 3(1): 4-6.
11. Barrera-Figueroa, V., I. Guerrero-Moreno, J. López-Bonilla and S. Vidal-Beltrán, 2018. Some applications of hypergeometric functions, Comput. Appl. Math. Sci., 3(2): 23-25..
12. Pearson, J., 2009. Computation of hypergeometric functions, Master of Science Thesis, University of Oxford.
13. Koepf, W., M. Masjed-Jamei, 2006. A generic polynomial solution for the differential equation of hypergeometric type and six sequences of orthogonal polynomials related to it, Integral Transforms and Special Functions, 17(8): 559-576.