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Lanczos generator for the Weyl Tensor

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Abstract: We exhibit the Lanczos potential for the conformal tensor in arbitrary spacetimes of Petrov types O, N and III.

Key words: Weyl tensor · Lanczos generator · Petrov classification · Canonical null tetrad

INTRODUCTION

The Lanczos generator $K_{\mu\nu\alpha}$ [1-5] has the following algebraic symmetries:

$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}, \qquad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0, \qquad K^{\mu\nu}{}_{\nu} = 0, \tag{1}$$

and it generates the conformal tensor [6, 7] via the relation [8]:

$$C_{\mu\nu\alpha\beta} = K_{\mu\nu\alpha;\beta} - K_{\mu\nu\beta;\alpha} + K_{\alpha\beta\mu;\nu} - K_{\alpha\beta\nu;\mu} + \frac{1}{2}[\left(K_{\mu\beta} + K_{\beta\mu}\right)g_{\nu\alpha} + (K_{\nu\alpha} + K_{\alpha\nu})g_{\mu\beta} - \left(K_{\mu\alpha} + K_{\alpha\mu}\right)g_{\nu\beta} - \left(K_{\nu\beta} + K_{\beta\nu}\right)g_{\mu\alpha}], \qquad K_{\mu\nu} \equiv K_{\mu\sigma\nu}{}^{;\sigma}. \tag{2}$$

If we select [9]:

$$K_{\mu\nu\alpha} = \frac{1}{3}(2F_{\mu\nu;\alpha} + F_{\alpha\nu;\mu} - F_{\alpha\mu;\nu} + F_{\nu\lambda}{}^{;\lambda}g_{\alpha\mu} - F_{\mu\lambda}{}^{;\lambda}g_{\alpha\nu}), \tag{3}$$

for arbitrary $F_{\mu\nu} = -F_{\nu\mu}$, then it is simple to obtain the property:

$$*K_{\mu\nu\alpha} = \frac{1}{3}(2\ *F_{\mu\nu;\alpha} + *F_{\alpha\nu;\mu} - *F_{\alpha\mu;\nu} + *F_{\nu\lambda}{}^{;\lambda}g_{\alpha\mu} - *F_{\mu\lambda}{}^{;\lambda}g_{\alpha\nu}), \tag{4}$$

with the participation of the dual tensors $*K_{\mu\nu\alpha} = \frac{1}{2}\eta_{\mu\nu\lambda\beta}\ K^{\lambda\beta}{}_{\alpha}$ and $*F_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu\lambda\beta}\ \mathrm{F}^{\lambda\beta}$. Thus (3) and (4) are equivalent to:

$$S_{\mu\nu\alpha} = K_{\mu\nu\alpha} + i\ \ *K_{\mu\nu\alpha} = \frac{1}{3}(2S_{\mu\nu;\alpha} + S_{\alpha\nu;\mu} - S_{\alpha\mu;\nu} + S_{\nu\lambda}{}^{;\lambda}g_{\alpha\mu} - S_{\mu\lambda}{}^{;\lambda}g_{\alpha\nu}), \tag{5}$$

where $S_{\mu\nu} = F_{\mu\nu} + i\ *F_{\mu\nu}$.

The application of (5) into (2) gives the expression:

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$$S_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + i\ {}^{*}C_{\mu\nu\alpha\beta} = C_{\sigma\nu\alpha\beta}\ S^{\sigma}{}_{\mu} - C_{\alpha\mu\alpha\beta}\ S^{\sigma}{}_{\nu} + C_{\sigma\beta\mu\nu}\ S^{\sigma}{}_{\alpha} - C_{\sigma\alpha\mu\nu}\ S^{\sigma}{}_{\beta}, \quad (6)$$

which implies 0 = 0 for an arbitrary conformally flat space, that is, (3) is a Lanczos potential for any Petrov type O spacetime.

Now we consider the following Petrov types in the canonical null tetrad [6, 7, 10, 11]:

a). Type N.

$$S_{\mu\nu\alpha\beta} = \psi_4\ V_{\mu\nu} V_{\alpha\beta}, \qquad V_{\mu\nu} = l_\mu m_\nu - l_\nu m_\mu, \qquad M_{\mu\nu} = m_\mu \overline{m}_\nu - m_\nu \overline{m}_\mu + n_\mu l_\nu - n_\nu l_\mu, \quad (7)$$

$$S_{\mu\nu} = q\ M_{\mu\nu}, \qquad F_{\mu\nu} = q(n_\mu l_\nu - n_\nu l_\mu) \qquad {}^{*}F_{\mu\nu} = -iq(m_\mu \overline{m}_\nu - m_\nu \overline{m}_\mu), \quad (8)$$

then the NP components of (6) imply $\psi_4 = 2q\ \psi_4$, therefore (3) is a Lanczos potential with (8) for $q = \frac{1}{2}$.

b). Type III.

$$S_{\mu\nu\alpha\beta} = \psi_3 (V_{\mu\nu} M_{\alpha\beta} + M_{\mu\nu} V_{\alpha\beta}), \quad (9)$$

hence (6), (8) and (9) give $\psi_3 = q\psi_3$, thus (3) is a Lanczos generator with (8) for $q = 1$.

The Lanczos potential for arbitrary Petrov types III and N geometries has the structure (3) if we use the corresponding canonical null tetrad and $F_{\mu\nu}$ is given by (8) with $q = 1$ and $q = \frac{1}{2}$, respectively; in Petrov type O spacetimes, we can use (3) with any $F_{\alpha\beta}$. The construction of $K_{\mu\nu\alpha}$ for arbitrary 4-spaces of types D, II and I, is an open problem.

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