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On a Result for the Terminating $_2F_1(1)$

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Abstract: We obtain the value of $_2F_1\left(\frac{k-n}{2},\frac{k-n+1}{2};k+1;1\right),\quad 0\le k\le n,$ which is applied to one combinatorial identity studied by Nemes *et al*.

Key words: Gauss hypergeometric function, Binomial coefficients.

INTRODUCTION

The Gauss hypergeometric function [1-4] satisfies the following property [5]:

$$_2F_1\left(\frac{a}{2},\frac{a+1}{2};a-b+1;x\right)=2^a(1+\sqrt{1-x})^{-a}\ {}_2F_1\left(a,b;\ a-b+1;\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right), \tag{1}$$

which for $a = k - n$, $b = -n$, $x = 1$ implies the relation:

$$_2F_1\left(\frac{k-n}{2},\frac{k-n+1}{2};k+1;1\right)=2^{k-n}\ {}_2F_1\left(k-n,-n;\ k+1;1\right), \tag{2}$$

On the other hand, we have the Lanczos identity [6]:

$$\frac{1}{p};\ {}_2F_1(-x,-n;\ p+1;1)=\frac{(x+p+n)!}{(x+p)!(n+p)!}=\sum_{r=0}^{\infty}\frac{r!}{(r+p)!}\binom{n}{r}\binom{x}{r},\quad \forall x,p\ \ \&\ \ n=0,1,2,... \tag{3}$$

Where we can employ $x = n - k$, $p = k$, with $0 \le k \le n$, to obtain the result:

$$_2F_1(k-n,-n;k+1;1)=\frac{(2n)!k!}{n!(n+k)!}=\sum_{r=0}^{n-k}\frac{r!}{(r+k)!}\binom{n}{r}\binom{n-k}{r}, \tag{4}$$

Whose application in (2) gives the following value:

$$_2F_1\left(\frac{k-n}{2},\frac{k-n+1}{2};k+1;1\right)=\frac{\binom{2n}{n+k}}{2^{n-k}\binom{n}{k}},\quad 0\le k\le n. \tag{5}$$

For $k = 0$ the relation (4) implies the identity [7-10]:

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$$\sum_{r=0}^{n}\binom{n}{r}^2 = \frac{(2n)!}{(n!)^2} = {}_2F_1(-n,-n;1;1). \tag{6}$$

Nemes *et al*. [11, 12] deduced the expression:

$$A \equiv \sum_{j=0}^{n} 2^{n-k-2j}\binom{n}{j}\binom{n-j}{j+k} = \binom{2}{n+k}, \quad 0 \le k \le n, \tag{7}$$

and it is easy to obtain its hypergeometric version, in fact:

$$A \equiv 2^{n-k}\binom{n}{k}\sum_{r=0}^{\infty} t_r, \quad t_r = \frac{\binom{n}{r}\binom{n-r}{r+k}}{2^{2r}\binom{n}{k}} \quad \therefore \quad \frac{t_{r+1}}{t_r} = \frac{\left(r+\frac{k-n}{2}\right)\left(r+\frac{k-n+1}{2}\right)}{(r+k+1)\ (r+1)}, \tag{8}$$

Therefore [7, 8, 13, 14]:

$$A = 2^{n-k}\binom{n}{k}\ {}_2F_1\left(\frac{k-n}{2},\frac{k-n+1}{2};k+1;1\right) \overset{(5)}{=} \binom{2n}{n+k}, \tag{9}$$

In harmony with (7).

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