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# On the Variation of Parameters Method

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**Abstract:** If for a linear differential equation of second order (LDE) we know one solution of its associated homogeneous equation (HE), then other independent solution of HE and the particular solution of LDE are obtained via the Riccati equation. Our method is an alternative to the Lagrange 'ansatz'.

**Key words:** Riccati equation · Variation of parameters · Linear differential equation of 2nd order

## INTRODUCTION

If for the differential equation [1-4]:

$$p(x)y''+q(x)y'+r(x)y=\phi(x), \tag{1}$$

We know the solution $y_1(x)$ of its associated homogeneous equation [5-6]:

$$p\,y''+q\,y'+r\,y=0, \tag{2}$$

Then the other independent solution of (2) is given by [1-6]:

$$y_2(x)=y_1(x)\int^x \frac{w(\xi)}{[y_1(\xi)]^2}d\xi, \tag{3}$$

Where the wronskian [7] $W$ of $y_1$ and $y_2$ obeys the Abel-Liouville-Ostrogradski relation [2]:

$$W(y_1,y_2)=\exp(-\int^x \frac{q}{p}d\xi). \tag{4}$$

Hence the general solution of (1) adopts the form:

$$y(x)=c_1y_1+c_2y_2+y_p, \tag{5}$$

With $y_p$ verifying (1). The variation of parameters technique [1, 8, 9], introduced by Newton (Principia), Euler (1741) and Lagrange (1759), searchs a particular solution with the structure $y_p = A(x)\, y_2 + B(x)\, y_1$ and it obtains the expression [10]:

$$y_p(x)=y_2(x)\int^x \frac{y_1\phi}{pW}d\xi - y_1(x)\int^x \frac{y_2\phi}{pW}d\xi, \tag{6}$$

In this work we employ the procedure of Euler [11-14] to transform (2) into a Riccati equation [13-18], which generates a simple method to deduce (3) and (6) without the Lagrange approach.

The function $y_1$ satisfies $py_1''+qy_1'+ry_1=0$ where we can realize the Euler transformation [11]:

$$y_1(x)=\exp\left(-\int^x R_1(t)dt\right), \tag{7}$$

To obtain the Riccati equation:

$$R_1'-R_1^2+\frac{q}{p}R_1=\frac{r}{p}. \tag{8}$$

Now we introduce the function $\eta(x)$ such that:

$$R_1=\frac{1}{y}(\eta-y'), \tag{9}$$

With $y(x)$ verifying (1); therefore (8) and (9) imply two linear differential equations of first order:

$$\eta'+\left(\frac{q}{p}-R_1\right)\eta=\frac{\phi}{p}, \tag{10}$$

$$y'+R_1y=\eta. \tag{11}$$

By hypothesis we know $y_1$, that is, $R_1$ via (7), then it is immediate the solution of (10):

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$$\eta(x) = \frac{W}{y_1}\int^x \frac{y_1\phi}{p^w}d\xi + c_2\frac{W}{y_1}, \qquad (12)$$

Which allows to solve (11):

$$y(x) = y_1\int^x \frac{\eta}{y_1}d\tau + c_1 y_1; \qquad (13)$$

With (12) into (13) we obtain (5) where $y_2\,(x)$ is given by (3) and:

$$y_p(x) = y_1\int^x \frac{W(\xi)}{[y_1(\xi)]^2}d\xi \int^\xi \frac{y_1(\tau)\phi(\tau)}{p(\tau)W(\tau)}d\tau = y_1(x)\int^x u\ dv, \qquad (14)$$

such that $u = \int^\xi \frac{y_1\phi}{pW}d\tau$ & $dv = \frac{W(\xi)}{[y_1(\xi)]^2}d\xi.$ If in (14) we apply integration by parts we deduce (6), the basic relation of the variation of parameters method.

Our process, based in the Riccati equation, shows an alternative approach to the Lagrange 'ansatz'.

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