American-Eurasian Journal of Scientific Research 11 (6): 465-468, 2016
ISSN 1818-6785

DOI: 10.5829/idosi.aejsr.2016.465.468

Stability Analysis of a Ratio-Dependent Predator-Prey Interactions with Epidemic in the Prey

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Abstract: A ratio-dependent predator-prey model with disease in the prey population is formulated and analyzed. Assuming that prey populations suffered from epidemics and the predator population will prefer only infected population for their diet as those are more vulnerable. Dynamical behaviours such as boundedness, permanence, local and global stabilities are discussed.

Subject Classification: 92B05
Key words: Global stability・Epidemiological model・Hopf bifurcation

INTRODUCTION

The dynamic relationship between predator and their prey has long been and will continue to be one of the dominant topics in both applied mathematics and theoretical ecology due to its universal existence and importance. These problems may appear to be simple mathematically at first sight, they are, in fact, often very challenging and complicated.

Since the pioneering work of Kermack-Mckendrick on SIRS [1], epidemiological models have received much attention from scientists. Mathematical models have become important tools in analyzing the spread and control of infectious disease. It is of more biological significance to consider the effect of interacting species when we study the dynamical behaviors of epidemiological models. Eco epidemiology which is a relatively new branch of study in theoretical biology, tackles such situations by dealing with both ecological and epidemiological issues. It can be viewed as the coupling of an ecological predator-preymodel and an epidemiological SI, SIS, or SIRS model. Following Anderso and May [2] who were the first to propose an eco-epidemiological model by merging the ecological p redator-prey model introduced by Lotka and Volterra, the effect of disease in ecological system is an important issue from mathematical and ecological point of view. Many works have been devoted to the study of the effects of a disease on a predator-prey system [1-6]. In this paper, the dynamical behaviourof a ratio-dependent predator-prey systems with infection in prey populationis investigated. Here we have studied the boundedness, permanence, local and global stabilities of the non-equilibrium points of this system.

The Mathematical Model:

$$\begin{cases} \frac{dS}{dt} = r_1(1-\frac{S+I}{K})S-\beta SI \\ \frac{dI}{dt} = \beta SI - dI - \frac{bIY}{aY+I} \\ \frac{dY}{dt} = -cY + \frac{pbIY}{aY+I} \end{cases} \tag{1}$$

With initial data $S(0) \geq 0$, $I(0) \geq 0$, $Y(0) \geq 0$,

They make the following assumptions in formulating the mathematical model of apredator-prey system with disease in the prey population:

- In the absence of disease, the prey population grows logistically with carrying capacity $K \in \Re_+$ and intrinsic birth rate $r \in \Re$.
- In the presence of virus, the prey population is divided into two groups, namely susceptible prey denoted by S(T) and infected prey denoted by I(T). Therefore at time T, the total population is N(T) = S(T) + I(T).
- The disease is not genetically inherited. The infected populations do not recover or become immune. We assume that the disease transmission follows the simple law of mass action.

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β S(T)I(T) with β as the transmission rate.

- The infected prey I(T) is removed by death (say, its death rate is positive constant c) or by predation before having the possibility of reproducing. However, the infected prey population I(T) still contribute with S(T) towards the carrying capacity of the system.
- The infected prey is more vulnerable than susceptible prey. We assume that thepredator population consumes only infected prey with ratio-dependent Michaelis–Menten functional response function.

$$\mu(I,Y)=\frac{IY}{aY+I},\ (a>0)$$

It is assumed that the predator has the death rate constant d (c> 0) and the predation coefficient b (b > 0). The coefficient in conversing prey into predator is $p(0 < p \le 1)$.

To reduce the number of parameters and to determine which combinations of parameters control the behaviour of the system, we nondimensionalize system (2). We choose.

$$\begin{aligned}\frac{ds}{dt}&=r(1-(s+i))s-\beta si\\ \frac{di}{dt}&=\beta si-ei-\frac{qiy}{y+i}\\ \frac{dy}{dt}&=-wy+\frac{pqiy}{y+i}\end{aligned}\qquad (2)$$

With initial data $s(0)\ge 0$, $i(0)\ge 0$, $y(0)\ge 0$

where $r=\frac{r_r}{\beta K}, w=\frac{d}{\beta K}, q=\frac{b}{a\beta K}, e=\frac{c}{a\beta K}$

Boundedness

Theorem 3.1: Any solution of system (2) is uniformly bounded in $\Re_+^3$.

Proof: Let $(s(t), i(t), y(t))$ be any solution of the system (2). Since, $\frac{ds}{dt}=rs(1-s)$

We have, $\limsup_{t\to\infty} s(t)\le r$

$V=\frac{s}{1+r}+i+y$. Then,

$$\frac{dV}{dt}=\frac{r}{1+r}s(1-s)-wi-\frac{e}{p}y\le\frac{r}{1+r}s-wi-\frac{e}{p}y$$

$$\frac{dV}{dt}\le\frac{2r}{1+r}-\eta V$$; where $\eta = \min\{1, w, e\}$

Therefore, $\frac{dV}{dt}+\eta V\le\frac{2r}{1+r}$. Applying theorem on differential inequalities [7], we obtain.

$0\le V(s,i,y)\le\frac{2r}{(1+r)\eta}+\frac{V(s(0),i(0),y(0))}{e^{\eta t}}$ and as $t\to\infty$ $0\le V\le$, $\frac{2r}{(1+r)\eta}$

Thus, all the solution of (2) enter into the region,

$$D=\left\{(s,i,y):0\le V\le\frac{2r}{(1+r)\eta}+\varepsilon\ for\quad any\quad \varepsilon>0\right\}$$

Hence the theorem holds.

Equilibrium Points and Stability Analysis: The equilibrium points are obtained by solving $\frac{ds}{dt}=\frac{di}{dt}=\frac{dy}{dt}=0$ It is found that the system (2) has two boundary equilibrium $E_0(0, 0, 0)$, the axial equilibrium $E_1(1, 0, 0)$, the predator-free equilibrium point $E_2(\bar{s},\bar{i},0)$ where $\bar{s}=w$ and $\bar{i}=\frac{r(1-w)}{1+r}$ and the interior equilibrium $E^*(s^*, i^*, y^*)$.

where $s^*=\frac{pw+(pq-e)}{r}$, $i^*=\frac{r}{p(1+r)}\left(p(1\,{}^{p}\,w)-(pq-e)\right)$,

and $y^*=\frac{r(pq-e)}{ep(1+r)}\left(p(1-w)-(pq-w)\right)$

The system (2) cannot be linearized at $E_0(0, 0, 0)$ and $E_1(1, 0, 0)$ and therefore local stability of E_0 and E_1cannot be studied [8].

LEMMA 4.1: The predator-free equilibrium point $E_2(\bar{s},\bar{i},0)$ exists if and only if $w < 1$

The Jacobean Matrix at the equilibrium point E_2 is given by;

$$J(E_2)=\begin{pmatrix}-r\bar{s} & -(1+r)\bar{s} & 0\\ \bar{i} & 0 & -q\\ 0 & 0 & -e+pq\end{pmatrix}$$

The characteristics equation of $J(E_2)$ is $(\lambda^2+B\lambda+C)(\lambda-pq+e)=0$,

where $B=r\bar{s}>0$ and $C=(1+r)\bar{s}\bar{i}>0$.

The eigenvalues are $\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$ and $\lambda_3 = qp - e$

Since, $B > 0$ and $C > 0$, therefore, the signs of the real parts λ_1 and $\lambda 2$ E_2 is locally asymptotically stable in the si-plane. Now E_2 is asymptotically stable in the y-direction if and only if $pq - e < 0$

LEMMA 4.2: The interior equilibrium $E^*(s^*, i^*, y^*)$ of the system (2) existsif and only if the following conditions hold:

(a) $pq > e$
(b) $p(1 - w) - (pq - e) > 0$

Which are the necessary and sufficient conditions for the co-existence of the susceptible prey, infected prey and the predator.

Local Stability of E^*

$$J(E^*) = \begin{pmatrix} j_{11} & j_{12} & 0 \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & j_{33} \end{pmatrix} \text{ where}$$

$$j_{11} = -rs^*, j_{12} = -(1+r)s^*, j_{21} = i^*, j_{22} = \frac{qi^* y^*}{(i^* + y^*)^2},$$

$$j_{23} = \frac{qi^{*2}}{(i^* + y^*)^2}, j_{32} = \frac{pqy^{*2}}{(i^* + y^*)^2}, j_{32} = -\frac{pqi^* y^*}{(i^* + y^*)^2}$$

The characteristics is $\lambda^3 + a_1 \lambda^2 + a_2\lambda + a_3 = 0$

$$a_1 = -trJ(E^*) = -j_{11} - j_{22} - j_{33} = rs^* - \frac{q(1-p)i^* y^*}{(y^* + i^*)^2}$$

$$a_1 = \frac{\Gamma}{qp^2}; \Gamma = rqp^2 w + pe(pq - e)(pq - e)rpq - e)$$

$$a_2 = j_{11}j_{22} + j_{11}j_{33} + j_{22}j_{33} - j_{23}j_{32} - j_{12}j_{21}$$

$$= i^* s^* \left\{ (1+r) - \frac{rq(1-p)y^*}{(y^* + i^*)^2} \right\}$$

$$a_3 = -\det[J(E^*)] = j_{11}j_{23}j_{32} + j_{12}j_{21}j_{33} - j_{11}j_{22}j_{33}$$

$$= \frac{pq(1+r)s^* y^* i^{*2}}{(y^* + i^*)^2}$$

Now,

$$\Delta = a_1 a_2 - a_3$$

$$= -(j_{11})^2 j_{22} - (j_{11})^2 j_{33} + j_{11}j_{12}j_{21} - (j_{22})^2 j_{33} - (j_{22})^2 j_{11}$$

$$= -2j_{11}j_{22}j_{33} + j_{22}j_{12}j_{21} + j_{23}j_{32}j_{22} - j_{22}(j_{33})^2 - j_{11}(j_{33})^2 + j_{23}j_{32}j_{33}$$

$$\Delta = i^* s^* \left\{ (1+r)rs^* - \frac{r^2 q(1-p)s^* y^*}{(y^* + i^*)^2} + \frac{rq^2(1-p)^2 i^* y^{*2}}{(y^* + i^*)^4} - \frac{q(1+r)i^* y^*}{(y^* + i^*)^2} + \right\}$$

Theorem 4.3: E^* is locally asymptotically stable if and only if $\Gamma > 0$ and $\Delta > 0$

Proof: Note that

- $\Gamma > 0$ if and only if $a_i > 0$.
- $a_3 > 0$ for all value of the parameters. and
- $\Delta = a_1 a_2 - a_3 > 0$.

Hence, from Routh Hurwitz criterion the theorem holds.

Theorem 4.4: Existence of positive equilibrium of the system (2) implies its global stability around the positive interior equilibrium.

CONCLUSION

In this paper, we have studied an eco-epidemiological model with disease in the prey population which is governed by modified logistic equation. It is shown (in Theorem 3.1) that the non-dimensionalized system (2) is uniformly bounded, whichin turn, implies that the system is biologically well behaved. The condition for whichall three species will persist are worked out. In deterministic situation, theoretical epidemiologistsare usually guided by an implicit assumption that most epidemic modelswe observe in nature correspond to stable equilibria of the models. From this viewpoint, we have presented the most important equilibrium point $E^*(s^*, i^*, y^*)$. The stabilitycriteria given in Lemma 4.2 and Theorem 4.3 are the conditions for stable coexistence of thesusceptible prey population, infected prey population and predator population.

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