Studies in Nonlinear Sciences 3 (3): 86-93, 2012
ISSN 2221-3910

DOI: 10.5829/idosi.sns.2012.3.3.249

Numerical Solutions of Falkner-Skan Equation with Heat Transfer

[1]F. Mohammadi, [1]M.M. Hosseini, [2]A. Dehgahn and [1]F.M. Maalek Ghaini

[1]Faculty of Mathematics, Yazd University, P.O. Box: 89195-74, Yazd, Iran
[2]School of Mechanical Engineering, Yazd University, Yazd, Iran

Abstract: In this paper, the shifted Legendre polynomials are employed for solving the Falkner-Skan equation with heat transfer. The operational matrices of derivatives for shifted Legendre polynomial are derived. Then the shifted Legendre polynomials expansions along with these operational matrices are applied for solving Falkner-Skan equations with heat transfer. The advantage of the proposed method is that the need guess of the velocity slope, $f''(0)$, is dismissed. Moreover, by using the given boundary conditions in the problem, a stable solution with very good results can be obtained. The proposed method is evaluated by solving kinds of Falkner-Skan equations. Numerical results show higher efficiency and performance of the presented method in comparison with the conventional shooting method.

AMS MSC 2000: 65T60 . 65P99 . 65L10

Key words: Falkner-Skan equation . legendre polynomials . operational matrix . collocation method

INTRODUCTION

The Falkner-Skan equation arises in the study of laminar boundary layers flow exhibiting similarity solution. Assuming steady, incompressible, laminar flow with constant fluid properties and negligible viscous dissipation, the boundary layer equations can be reduced to [1]

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_e\frac{\partial U_e}{\partial x}+v\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where $U_e(x)$ is the free stream velocity, u and v are the velocity components in x and y directions respectively and v is the kinematic viscosity. Solution of these equations is simplified by the fact that for constant properties, conditions in the velocity boundary layer are independent of temperature species concentration. Hence we may begin by solving the hydrodynamic problem (1) and (2) to the exclusion of Eq. (3). Once the hydrodynamic problem has been solved, solution to equation (3) can be obtained. In the particular case of the two-dimensional, incompressible boundary-layer flow over a wedge, when the free stream velocity is of the form $U_e(x) = Kx^m$, the governing partial differential equations can be converted to ordinary differential equation by employing the following similarity transformation:

$$u(x,y)=U_e(x)f(\eta), \quad \eta=\sqrt{\frac{(m+1)}{2}\frac{K}{v}}x^{\frac{m-1}{2}}$$

Corresponding Author: F. Mohammadi, Faculty of Mathematics, Yazd University, P.O. Box: 89195-74, Yazd, Iran

This leads Esq. (1) and (2) to the well known Falkner-Skan equation

$$\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} + \beta\left(1 - \left(\frac{df}{d\eta}\right)^2\right) = 0 \tag{4}$$

with the following boundary conditions

$$f(0) = 0$$
$$f'(0) = 0 \tag{5}$$
$$f'(\eta = \infty) = 1$$

The condition $f'(\eta = \infty) = 1$ is usually replaced by the condition $f'(\eta_\infty) = 1$ for some sufficiently large η_∞. By this transformations, the energy equation, i.e. Eq. (3) is transformed into

$$\frac{d^2\vartheta}{d\eta^2} + \Pr f\frac{d\vartheta}{d\eta} = 0 \tag{6}$$

with the following boundary conditions

$$\vartheta(0) = 1,\ \vartheta(\eta_\infty) = 0 \tag{7}$$

In Eq. (6) Pr is the ratio of the kinematic viscosity to the thermal diffusivity of the fluid and is considered to be constant. The Falkner-Skan equation constitutes a third-order, nonlinear two point boundary-value problem, no exact analytical solution is known. In the case of $\beta = 0$, the Falkner-Skan equation reduces to the well known Blasius equation which is perhaps one of the most famous equations of fluid dynamics and represents the problem of an incompressible fluid that passes on a semi-infinity flat plate. In the case of accelerating flows (β>0), the velocity profiles have no points of inflection, whereas in the case of decelerated flows (β<0). Physically relevant solutions exist only for $-0.19884<\beta\leq 2$ [2].

Several methods have been used for numerical solution of Falkner-Skan equations. Meksyn [3] solved the Falkner-Skan equation through analytical approximations. Asaithambi [4, 5] used finite difference method and piecewise linear functions for solving Falkner-Skan equations with high accuracy. Recently Shi-Jun Liao [6] applied the homotopy analysis to solve the Falkner-Skan equation. Khabibrakhmanov and D. Summers [7] used a spectral method with generalized Laguerre polynomials for solving the Blasius equation ($\beta = 0$). Moreover, The Blasius equation was solved by Rosales and Valencia [8] using Fourier series. Also, Vera and Valencia [9] solved the Falkner-Skan equation with heat transfer through an expansion in Fourier series. In this paper, the shifted Legendre polynomials and their operational matrices are applied for numerical solution of both Falkner-Skan equation and its heat transfer equation. The advantage of the proposed method is that the need to guess the initial slope $f''(0)$ is dismissed and therefore the Falkner-Skan equation can be directly solved by solely employing the given boundary conditions (5).

SPECTRAL METHODS

Since the time of Fourier (1882), orthogonal functions and polynomials in analytic study of differential equations are used and their applications for numerical solution of ordinary differential equations refer, at least, to the time of Lanczos [10]. Moreover, origin of some current spectral (such as Galerkin, tau and pseudospectral) methods can be found in "weighted residual method" of Finlayson and Scriven [11].

It is well known that the eigenfunctions of certain singular Sturm-Liouville problems (such as Legendre or Chebyshev orthogonal polynomials) allow the approximation of functions $C^\infty[a,b]$ where truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation, as that number (order of truncation N) tends to infinity [12].

Shifted legendre polynomials and their properties: The well-known Legendre polynomials are defined on the interval [-1,1] and can be determined with the aid of the following recurrence formulae [12-14]

$$(m+1)L_{m+1}(t)=(2m+1)tL_m(t)-mL_{m-1}(t),\quad m=1,2,3,\ldots$$

where $L_0(t)=1, L_1(t)=t$. In order to use Legendre polynomials on the interval [0,1] we define the so-called shifted Legendre polynomials by introducing the change of variable t = 2x-1. Let the shifted Legendre polynomials $L_m(2x-1)$ are denoted by $P_m(x)$. Then $P_m(x)$ can be obtained as follows:

$$(m+1)P_{m+1}(x)=(2m+1)(2x-1)P_m(x)-mP_{m-1}(x),\quad m=1,2,3,\ldots$$

where $P_0(x) = 1$ and $P_1(x) = 2x-1$.
The orthogonality condition for these polynomials is

$$\int_0^1 P_m(x)P_n(x)dx=\begin{cases}\dfrac{1}{2m+1} & \text{for } m=n\\ 0 & \text{for } m\neq n\end{cases}$$

In the next theorem we derived a relation between shifted Legendre wavelets and their derivatives that is very important for deriving the operational matrix of derivative for Legendre wavelet.

Theorem 1: Let $P_m(x)$ be the shifted Legendre polynomials into [0,1] and $P'_m(x)$ be derivative of $P_m(x)$ with respect to x, then we have

$$P'_m(x)=2\sum_{\substack{k=0\\ k+m\ \text{odd}}}^{m-1}(2k+1)P_k(x) \tag{8}$$

Proof: Suppose the Legendre expansion of function u(x) be as

$$u(x)=\sum_{k=0}^{\infty}\hat{u}_k L_k(x) \tag{9}$$

then $u'(x)$ can be represented as [12]

$$u'(x)=\sum_{k=0}^{\infty}\hat{u}_k^{(1)}L_k(x) \tag{10}$$

where

$$\hat{u}_k^{(1)}=(2k+1)\sum_{\substack{p=k+1\\ p+k\ \text{odd}}}^{\infty}\hat{u}_p,\quad k\geq 0 \tag{11}$$

Now, by taking $u(x) = L_m(x)$ in Eq. (10) we have $\hat{u}_m=1$ and $\hat{u}_i=0$ for $i\neq m$, consequently

$$\hat{u}_k^{(1)}=\begin{cases}2k+1 & m+k \text{ is odd},\ k\leq m-1\\ 0 & \text{otherwise}\end{cases}$$

As a result Eq. (8) becomes

$$L_m'(x)=\sum_{\substack{k=0\\ m+k\ \text{odd}}}^{m-1}(2k+1)L_k(x) \tag{12}$$

By substituting x = 2t-1 in Eq. (12) we have

$$P'_m(t)=2\sum_{\substack{k=0\\ m+k\ \text{odd}}}^{m-1}(2k+1)P_k(t) \tag{13}$$

this proves the desired result.

Theorem 2: Let $\Psi(t)$ be the Legendre polynomial vector defined as

$$\Psi(t)=[P_0(t),P_1(t),\ldots,P_M(t)]$$

the derivative of this vector can be expressed by

$$\frac{d\Psi(t)}{dt}=D\Psi(t) \tag{14}$$

which F is $(M+1)\times(M+1)$ matrix and its (i,j)-th element is defined as below

$$D_{i,j}=\begin{cases}2(2j-1) & j=1,\ldots,i-1 \text{ and } (i+j) \text{ odd}\\ 0 & \text{othewise}\end{cases} \tag{15}$$

Proof: By using shifted Legendre polynomial into [0,1] the i-th element of vector $\Psi_i(t)$ in Eq. (14) can be written as

$$\Psi_i(t)=P_{i-1}(t) \tag{16}$$

By differentiation with respect to t in (16) we have

$$\frac{d\Psi_i(t)}{dt}=P'_{i-1}(t) \tag{17}$$

Now by substituting $P'_{i-1}(t)$ from Eq. (13) into (17) we get

$$\frac{d\Psi_i(t)}{dt}=2\sum_{\substack{j=0\\ j+odd}}^{i-2}(2j+1)P_j(x) \tag{18}$$

This equation can be expanded in Legendre wavelets as

$$\frac{d\Psi_i(t)}{dt}=2\sum_{\substack{j=1\\ j+m\ odd}}^{i-1}(2j-1)\Psi_j(x) \tag{19}$$

From (14) we conclude that

$$D_{i,j}=\begin{cases}2(2j-1) & j=1,\ldots,i-1 \text{ and } (i+j) \text{ odd}\\ 0 & \text{othewise}\end{cases}$$

and this leads to desired results.

Corollary: By using Eq. (14) the operational matrix for n-th derivative can be derived as

$$\frac{d^n\Psi(x)}{dx^n}=D^n\Psi(x) \tag{20}$$

where D^n is the n-th power of matrix D.

ANALYSIS OF THE METHOD

Consider the Falkner-Skan equation (4) and the energy equation (6) with their revelent boundary conditions (5) and (7). With the change of variable $g=\frac{f}{\eta_\infty}$, $\theta=\frac{\vartheta}{\eta_\infty}$ and under transformation $t=\frac{\eta}{\eta_\infty}$ the Falkner-Skan equation (4) and (6) is transformed to

$$g''' + \eta^2_\infty g g'' + \eta^2_\infty \beta(1-(g')^2) = 0 \tag{21}$$

$$\theta'' + \Pr \eta^2_\infty g\theta' = 0 \tag{22}$$

and the boundary conditions (5) and (7) are transformed to

$$g(0) = g'(0) = 0, \quad g'(1) = 1 \tag{23}$$

$$\theta(0) = \frac{1}{\eta_\infty}, \ \theta(1) = 0 \tag{24}$$

We consider the expansion of the unknown function g(t) in (21), by shifted Legendre polynomial into interval [0,1] as:

$$g(t) = \sum_{i=0}^{\infty} c_i P_i(t)$$

In practice we only consider (M+1)-terms shifted Legendre polynomials. So, we have

$$g(t) = \sum_{i=0}^{M} c_i P_i(t) = C^T\Psi(t) \tag{25}$$

where $\Psi(t)$ and C are the Legendre polynomial and the shifted Legendre coefficient vector as below

$$\begin{aligned} &C = [c_0, c_1, \ldots, c_M]^T \\ &\Psi(t) = [P_0(t), P_1(t), \ldots, P_M(t)] \end{aligned} \tag{26}$$

By using the operational matrix D derived in (14) we have

$$\begin{aligned} &g'(t) = C^T D\Psi(t) \\ &g''(t) = C^T D^2\Psi(t) \\ &g'''(t) = C^T D^3\Psi(t) \end{aligned} \tag{27}$$

Substituting Eqs (25) and (27) into (21), we get

$$C^T D^3\Psi(t) + \eta^2_\infty \left(C^T\Psi(t)\right)\left(C^T D^2\Psi(t)\right) + \eta^2_\infty \beta(1-\left(C^T D\Psi(t)\right)^2) = 0 \tag{28}$$

Also, according to boundary conditions (23) we have

$$\begin{aligned} &g(0) = C^T\Psi(0) = 0 \\ &g'(0) = C^T D\Psi(0) = 0 \\ &g'(1) = C^T D\Psi(1) = 1 \end{aligned} \tag{29}$$

To find the solution g(t), we first collocate Eq. (28) at (M-2) points. For choosing suitable collocation points, we use the first (M-2) roots of shifted Legendre $P_{M+1}(t)$. These equations together with Eq. (29) generate (M+1) nonlinear equations which can be solved for unknown vector C using Newton's iterative method. After finding C the function g(t) given in Eq. (25) can be calculated.

By substituting the derived function g(t) in (22) we have a linear boundary value problem for the function θ(t). Similar to g(t) we approximate θ(t) in the terms of shifted Legendre wavelets as

$$\theta(t) = \sum_{i=0}^{M} b_i P_i(t) = B^T \Psi(t) \tag{30}$$

where

$$B = [b_0, b_1, \ldots, b_M]^T$$

By using the operational matrix D we have

$$\theta'(t) = B^T D \Psi(t)$$
$$\theta''(t) = B^T D^2 \Psi(t) \tag{31}$$

Substituting (30) and (31) in (22) we have

$$B^T D^2 \Psi(t) + \Pr \eta_\infty^2 C^T \Psi(t) B^T D \Psi(t) = 0 \tag{32}$$

Collocating Eq. (32) at the first (M-1) roots of shifted Legendre $P_{M+1}(t)$, we have (M-1) linear systems for unknown vector B. These linear systems together with boundary conditions (24) generate (M+1) linear equations which can be solved for unknown vector B using Newton's iterative method. By inserting obtained vector B in (30) the function $\theta(t)$ can be calculated.

NUMERICAL EXPERIMENTS

In this section, we apply the shifted Legendre method described in Section 3 for solving Falkner-Skan equation along with the energy equation. Here we set $\eta_\infty = 5$ and all results are derived with Maple 12 with 20 digits precision. In this method the initial condition, $f''(0)$, which is an indication of surface friction coefficient, can be obtained in a shifted Legendre polynomials series as

$$f''(0) = \frac{1}{\eta_\infty}\left(C^T D^2 \Psi(0)\right)$$

Table 1 shows the initial slope $f''(0)$ obtained by the shifted Legendre polynomials method for various values of β and for M=20.

After solving the Falkner-Skan Eq. (4) with the conditions (5) we substitute the derived function $f(\eta)$ in the energy equation (5). By solving the energy equation with its boundary conditions (6) the unknown function $\vartheta(\eta)$ can be obtained. In this way the unknown initial condition $\vartheta'(0)$ is derived as

$$\vartheta'(0) = B^T D \Psi(0)$$

The value of $\vartheta'(0)$ is the primary unknown of the similarity equations (4) to (7) and directly related to the convective heat transfer coefficient of the surface. Table 2 shows the initial slope $\vartheta'(0)$ obtained by the Legendre polynomials method for various values of β and Pr.

Table 1:

β	$f''(0)$
2	1.687217
1	1.232587
0.5	0.927601
0	0.469589
-0.12	0.281772
-0.15	0.216335
-0.18	0.128637

Table 2:

	$-\vartheta'(0)$		
β	Pr = 0.72	Pr = 6	Pr = 10
2	0.529661	1.177662	1.468978
1	0.501508	1.107140	1.317881
0.5	0.475721	1.038039	1.234244
0	0.418370	0.866649	1.030806
-0.15	0.367989	0.722827	0.849524

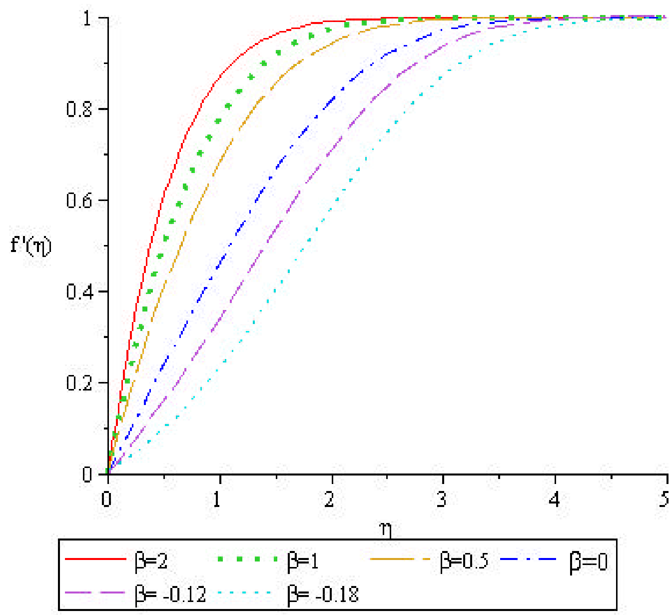

Fig. 1: Function $f'(\eta)$ different values of the parameter ß

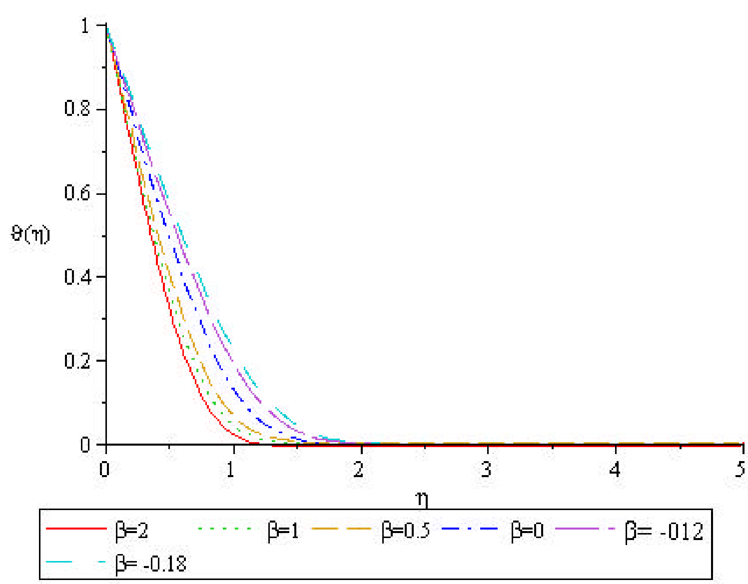

Fig. 2: Function $\vartheta(\eta)$ various values of the parameter ß and Pr=10

Figure 1 shows the variation of the non-dimensional velocity profile within the boundary layer, $f'(\eta)$, as a function of non-dimensional similarity variable η for different values of the parameter ß obtained with the expansion in Legendre polynomials.

Figure 2 shows the variation of the non-dimensional temperature profile, $\vartheta(\eta)$, as a function of non-dimensional similarity variable η for different values of the parameter ß and Pr=10.

CONCLUSION

The shifted Legendre polynomial and its operational matrix of derivatives are employed for solving nonlinear Falkner-Skan equation with its relevant energy equation. In the proposed method, the requirement of to guessing the initial condition $f''(0)$ in order to start the solution is dismissed. By using the given boundary conditions the method delivers a robust solution with very good accuracy.

ACKNOWLEDGEMENTS

This research project is supported by Intelligent Robust Systems Research Center of Yazd University.

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